MATH 5040/6810: Homework 2 (Due Monday, September 23)

Problem 1

Let N be an integer with $N \geq 4$. Consider a Markov chain with state space $\{1,\dots,N\}$. From 1 the chain moves to 3 or N equally likely. From a site j between 2 and $N-2$ the chain goes to $j-1$ or $j+2$ equally likely. From $N-1$ the chain jumps to 1 or $N-2$ equally likely and from N it goes to 2 or $N-1$ equally likely. Can you guess an invariant measure for the Markov chain? Explain your guess and then prove it is correct. Hint: Arrange the states 1 through N on a circle and draw the graphical representation of the Markov chain. What do you notice about the sites 1 through N?

Solution: Think of the above states as if they are N knights at a round table. Each knight is pointing the right index finger at the knight sitting two seats to his right and the left index finger at the knight sitting one seat to his left. Now you see that all sites are alike. Hence, the invariant measure should not distinguish between the different sites. In other words, it should be the uniform measure on N points. That is, $\Phi_\infty(j) = 1/N$ for all $j = 1,\dots,N$. Let us now check this is true.

The transition matrix is given by

$$P = \begin{bmatrix} 0 & 0 & 1/2 & 0 & 0 & 0 & \cdots & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & \cdots & \ddots & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & \ddots & \ddots & \ddots & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & \cdots & \cdots & 0 & 0 & 1/2 & 0 \end{bmatrix}.$$

Note that not only the rows add up to 1 but the columns too. Therefore, if π is the row vector with all entries equal to 1, then $\pi P = \pi$. But $\Phi_\infty = \frac{1}{N}\pi$. So $\Phi_\infty P = \Phi_\infty$ as well.

Problem 2

Consider the Markov chain on state space $\{a,b,c\}$. From a the chain goes to b with probability $1/3$ and to c with probability $2/3$. From b it goes to a or c equally likely. From c it goes to straight to a. Is this chain irreducible? What is its period? Compute the invariant measure by hand. Do not use a computer or a calculator for this problem.

Note: If you do not learn how to do this computation by hand on such a small example, you may struggle during exam(s)!

Solution: The chain is irreducible because it can go, with positive probability, from a to b to c and back to a. So all sites communicate. Since it is irreducible, the period is the same for all sites and we will choose to consider the site a. The chain is aperiodic (i.e. with period 1) because it can go back to a in 2 steps (e.g. a to b to a) and in 3 steps (e.g. a to b

to c to a). The greatest common divisor of 2 and 3 is 1 and so the period has to be 1 (since it is the greatest common divisor of all the possible numbers of steps with which one can return to a). The transition matrix is given by

$$P = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let the invariant measure be given by $\Phi_\infty = [x\ y\ z]$. Then $\Phi_\infty P = \Phi_\infty$ gives the equations

$$y/2 + z = x, \quad x/3 = y, \quad \text{and } 2x/3 + y/2 = z.$$

Note that adding the first two equations gives the third. So we can discard one of the three equations as being redundant. We choose to discard the first equation. We replace that by the equation saying that $x + y + z = 1$. Substitute y from the second equation into the third one to get that $z = 2x/3 + x/6 = 5x/6$. Substitute y and z into $x + y + z = 1$ to get

$$x + x/3 + 5x/6 = 1,$$

which gives $13x/6 = 1$ and hence $x = 6/13$ and from that $y = 2/13$ and $z = 5/13$. So the invariant measure is given by

$$\Phi_\infty = [6/13\ 2/13\ 5/13].$$

Problem 3

Consider the Markov chain on state space $\{4, 5, 6, 7\}$. From 4 the chain goes to 5 or 7 equally likely. From 5 it goes to 4 or 6 equally likely. From 6 it goes to 5 or 7 equally likely. From 7 it goes straight to 6. Is this chain irreducible? What is its period? Compute the invariant measure.

Solution: This Markov chain is irreducible because we can go, with positive probability, from 4 to 5 to 6 to 7 to 6 to 5 and return to 4, and hence all sites communicate with each other. Since the Markov chain is irreducible, all sites have the same period. We therefore can look at any of the sites and compute its period and the other sites will have the same period. We choose the site 7 because the only way to return to it is from 6. We observe that in order to return to 7 one needs to go to 6 and back (maybe multiple times). Similarly, if we are to return to 6 (before going on to 7), we need to go to 5 and back (again, maybe multiple times). The same argument is repeated for 5. So we see that to return to 7, the chain has to make an even number of steps. And since it can return in 2 steps, we see that the set of possible number of steps required to return to 7 (starting at 7) is all even numbers. The greatest common divisor of this set is 2, which is the period.

For the invariant measure, we repeat the ideas from the previous problem. The transition matrix is given by

$$P = \begin{bmatrix} 0 & 12 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

If we say $\Phi_\infty = [x\ y\ z\ t]$, then the equation $\Phi_\infty P = \Phi_\infty$ gives us four linear equations. Since P is stochastic, the equations are linearly dependent and we can ignore any one of them, replacing it with the equation $x+y+z+t=1$. We can solve these four equations by hand or by rewriting things as a linear algebra problem. Namely, $\Phi_\infty P = \Phi_\infty$ can be rewritten as $\Phi_\infty(P-I)=0$, where I is the identity matrix (ones on the diagonal and zeroes off the diagonal). Writing this out gives

$$[x\ y\ z\ t]\begin{bmatrix} -1 & 1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 & 0 \\ 0 & 1/2 & -1 & 1/2 \\ 0 & 0 & 1 & -1 \end{bmatrix} = [0\ 0\ 0\ 0].$$

Ignoring one of the equations means ignoring one of the columns of the matrix on the left-hand side and the corresponding column in the row vector of zeros on the right-hand side. We choose to ignore the last equation and so we get

$$[x\ y\ z\ t]\begin{bmatrix} -1 & 1/2 & 0 \\ 1/2 & -1 & 1/2 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = [0\ 0\ 0].$$

Replacing this equation we ignored by the equation $x+y+z+t=1$ means adding another column to the matrix, composed of all 1 entries, and also adding a 1 in the corresponding spot on the right-hand side. So we have

$$[x\ y\ z\ t]\begin{bmatrix} -1 & 1/2 & 0 & 1 \\ 1/2 & -1 & 1/2 & 1 \\ 0 & 1/2 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = [0\ 0\ 0\ 1].$$

Now we can multiply (on the right) by the inverse of the matrix to get

$$[x\ y\ z\ t] = [0\ 0\ 0\ 1]\begin{bmatrix} -1 & 1/2 & 0 & 1 \\ 1/2 & -1 & 1/2 & 1 \\ 0 & 1/2 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}^{-1}.$$

Plugging this into a software like Matlab gives

$$\Phi_\infty = [x\ y\ z\ t] = [1/8\ 1/4\ 3/8\ 1/4].$$

Problem 4

Consider the Markov chain on state space $\{2\}$. From 2 it goes back to 2. What is the invariant measure for this chain?

Solution: This is a very simple Markov chain with only a single element in its state space. The invariant measure, therefore, has only one entry. Since the entries have to add up to 1, this one entry is equal to 1. So $\Phi_\infty = [1]$.

Problem 5

Consider the Markov chain on state space $\{1,2,3,4,5,6,7,8\}$. From 1 it goes to 2 or 3 equally likely. From 2 it goes back to 2. From 3 it goes to 1, 2, 4, or 8 equally likely. From

4 the chain goes to 5 or 7 equally likely. From 5 it goes to 4 or 6 equally likely. From 6 it goes to 5 or 7 equally likely. From 7 it goes straight to 6. From 8 the chain goes back to 8.

(a) What are the communicating classes? Which are recurrent and which are transient? What are their periods?

Solution: The state 2 is absorbing and, therefore, forms its own recurrent class. Similarly for 8. The states $\{4, 5, 6, 7\}$ communicate with each other and form a single recurrent state because of a similar reasoning as in Problem 3. The states 1 and 3 communicate with each other and form one class. Since 3 points to 4, which is in a recurrent class (that does not point back to either 1 nor 3), $\{1, 3\}$ is a transient class. We only care about periods of recurrent classes. 2 and 8 have period 1 each because they point back to themselves. We already calculated the period of $\{4, 5, 6, 7\}$ in Problem 3 and found it to be 2.

(b) Since the transition matrix is stochastic we know it has an eigenvalue equal 1. What is the multiplicity of this eigenvalue?

Note: It is NOT a coincidence that multiplicity of 1 matches the number of recurrent communicating classes.

Solution: Asking Matlab for the eigenvalues of the matrix P we see that 1 is repeated three times. So it has multiplicity 3. We observe that this is the same as the number of recurrent classes.

(c) Use your answer to Problems 3 and 4 to give three different invariant measures for this Markov chain.

Solution: The three invariant measures are

$\Phi_\infty^{(1)} = [0\ 0\ 0\ 1/8\ 1/4\ 3/8\ 1/4\ 0]$, $\Phi_\infty^{(2)} = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$, and $\Phi_\infty^{(3)} = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 1]$.

(d) Prove that if you add a third of each of the invariant measures you found in (c) you will get again an invariant measure. What about half of one, a quarter of another, and a quarter of the third? How many invariant measures does this Markov chain have?!

Solution: We know that $\Phi_\infty^{(i)} P = \Phi_\infty^{(i)}$ and the entries of $\Phi_\infty^{(i)}$ add up to one, for each $i \in \{1, 2, 3\}$. If a, b, c are three nonnegative numbers that add up to 1 and we let

$$\Phi_\infty = a\Phi_\infty^{(1)} + b\Phi_\infty^{(2)} + c\Phi_\infty^{(3)},$$

then

$$\Phi_\infty P = a\Phi_\infty^{(1)} P + b\Phi_\infty^{(2)} P + c\Phi_\infty^{(3)} P = a\Phi_\infty^{(1)} + b\Phi_\infty^{(2)} + c\Phi_\infty^{(3)} = \Phi_\infty.$$

Furthermore, the sum of the entries of Φ_∞ equals a times the sum of the entries of $\Phi_\infty^{(1)}$ plus b times the sum of the entries of $\Phi_\infty^{(2)}$ plus c times the sum of the entries of $\Phi_\infty^{(2)}$, which is $a \cdot 1 + b \cdot 1 + c \cdot 1 = a + b + c = 1$. So Φ_∞ satisfies $\Phi_\infty P = \Phi_\infty$ and its entries add up to one. So it is an invariant measure. This answers all three parts of the question. The first part comes by setting $a = b = c = 1/3$. For the second part set $a = 1/2$, $b = 1/4$, and $c = 1/4$. And since there are infinitely many ways to choose nonnegative numbers a, b, c

that add up. to 1, the answer to the third part of the question is that there are infinitely many invariant measures.

(e) If you start the Markov chain at 1, what is the expected number of returns to 1? Compute this in two ways:

(e.1) By observing that from 1 you can go to 2, you can go to 3 then leave to 2, 4, or 8, or you can go to 3 then return to 1. With the first four moves you will never return to 1. Reduce this problem to a much simpler Markov chain (with only two states!) and find the mass function of the number of total times your return to 1, i.e. the probability it equals 0, 1, 2, etc. Now compute the average number of returns to 1.

Solution: The only way to return to 1 is to go to 3 and then back to 1. This has a probability of $1/2 \times 1/4 = 1/8$. The probability of not returning to 1 is then $7/8$ (which can be also seen by noting that we do not return to 1 by either going to 2 or going to 3 and then not taking the step back to 1, which has a probability of $1/2 + 1/2 \times (1 - 1/4) = 7/8$). Let us call "returning to 1" a failure and "not returning to 1" a success. Because of the Markov chain property, each time we return to 1 we start afresh. So we are looking for the average number of failures before the first success, in sequence of independent trials that give a failure with probability $1/8$ and a success with probability $7/8$. This is a geometric random variable, but one that counts the number of failures before a success, not the one that counts when the first success will occur. The expected value of the latter type is the reciprocal of the probability of success, which is $8/7$ here. But the expected value of the former (and the one we are interested in, since we are asked about the average number of returns) is one less than $8/7$, which is $1/7$. (This is because if, e.g., we get a success for the first time after 4 trials, then we had 3 failures.)

(e.2) Use the linear algebra method you learned in class. In this case, compute also the average number of visits to 3, starting at 1. Starting at 1, what is the average number of steps it takes before you leave the transient class forever?

Solution: Since 1 is a transient state, the answer to the problem is given by the appropriate entry of the matrix $M = (I - Q)^{-1}$. The matrix Q is the transient-to-transient submatrix of P, which is

$$Q = \begin{bmatrix} 0 & 1/2 \\ 1/4 & 0 \end{bmatrix}.$$

Thus,

$$M = (I - Q)^{-1} = \begin{bmatrix} 1 & -1/2 \\ -1/4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 8/7 & 4/7 \\ 2/7 & 8/7 \end{bmatrix}.$$

The average number of visits to 1, starting at 1, is $M_{1,1} = 8/7$. If we are after the number of returns to 1, we should subtract the first visit (since we start at 1) to get $1/7$, which is the same answer we obtained in (e.1). For the average number of visits to state 3, starting at 1, we get $4/7$. The total number of steps it takes to get out of the class $\{1, 3\}$, starting at 1, is the same as the total number of visits made to 1 plus the total number of visits made to 3. The average value is thus given by $8/7 + 4/7 = 12/7$.

(f) Starting at 1, what is the probability you will ever reach 2? Again, compute this in two ways:

(f.1) Let p be the probability we are after. Now observe that starting at 1 you can go to 2 directly, go to 3 then to 4 or 8 and thus never reach 2, go to 3 and then to 2, or go to 3 and back to 1 and now the story starts over again. Use this observation to write and solve a simple equation for p.

Solution: Using the law of total probability and the Markov property, we decompose "ever reaching 2" according to where the first step lands (after starting at 1). If we land at 2, then, after that, we reach 2 with probability 1. If we land at 3, then we either go after that to 2 and hence reach 2 with probability 1, or we go to 4 or 8 and from there we have a zero probability of reaching 2. Or, from 3, we jump back to 1 and the story restarts. So we see that we have the equation:

$$p = 1/2 + 1/2 \times (1/4 \times 1 + 1/4 \times 0 + 1/4 \times 0 + 1/4 \times p).$$

Simplifying, we get $p = 5/8 + p/8$ which gives $p = 5/7$.

(f.2) Use the linear algebra ideas you learned in class.

Solution: We collapse the recurrent states (which really just means collapsing $\{4, 5, 6, 7\}$ into one state we will call R, since 2 and 8 are already single recurrent states) and rearrange things so that the states come in the order: $R, 2, 8, 1, 3$. We thus get the transition matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}.$$

The matrices Q and M are as in (e.2). The matrix S is the transient-to-absorbing matrix and is given by

$$S = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

Then

$$MS = \begin{bmatrix} 1/7 & 5/7 & 1/7 \\ 2/7 & 3/7 & 2/7 \end{bmatrix}.$$

The probability we are looking for is that of starting at 1 and ending up at 1. So we look at the first row and the second column and get that the probability is $5/7$, as we found in (f.1).

(g) Starting at 1, what is the probability you will ever reach 8? (You could solve this in two ways too, but this time pick your favorite way of doing it.)

Solution: We will use linear algebra, since it is a more general method and does not depend on having a fortuitous structure that allows us to do things by hand. Also, we have already computed the required matrix MS in (f.2). We need to look at the first row (since we are starting at 1, not 3) and the third column (since we are ending at 8, not R nor 2), and we get that the probability we are after equals $1/7$.

(h) Starting at 4, what is the expected time of first return to 4?

Solution: Here, 4 is a recurrent state. So the average expected time of return to 4 is given by the reciprocal of the invariant measure at 4. But we need to be careful to use the invariant measure of the communication class of 4. We computed that in Problem 3. So the answer is $1/\Phi_\infty(4) = 8$.

(i) Starting at 4, what is the expected number of steps before you reach 5?

Solution: For this, we have to make 5 an absorbing state and compute a new Q matrix and then a new $M = (I - Q)^{-1}$ matrix. We can focus on the recurrent class $\{4, 5, 6, 7\}$ because starting at 4 we never leave that class. Since we make 5 absorbing, $\{4, 6, 7\}$ become transient and the transient-to-transient transition matrix becomes

$$Q = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

(The sites are in the order $4, 6, 7$ here.) Thus,

$$M = (I - Q)^{-1} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

Sanity check: the entries of the matrix are averages of numbers of visits. The diagonal entries are thus averages of numbers of visits from a site to itself, which means they should be no less than one. This is the case in our matrix. Furthermore, we notice that the average number of visits to 4, starting at 4, before we reach 5, is equal to 1. This means that once we leave 4 we never return to 4. Inspecting the Markov chain, we notice that indeed, once we leave 4 we are either at 5 and the game is over or we move to 7 and then there is no path back to 4. In any case, the answer to the question is that the expected number of steps before reaching 5 is equal to the sum of the number of visits to all the sites $4, 6, 7$, which is the sum of the first row of the matrix M (first row because we are starting at 4). The answer is therefore 3.

(j) What is the limit of $P\{X_n = 5 | X_0 = 1\}$ as $n \to \infty$?

Hint: You have computed the probability the chain ends up absorbed at 2 and at 8. This also gives you the probability the chain ends up in the communicating class containing 5. Also, in this particular chain, you know exactly where the chain enters that class. Once the chain is in that class and you give it a long time it reaches the equilibrium of that class, which you computed in part (c). Now answer the question. Be careful about the period, though.

Solution: First, observe that from 1 we can only enter the communicating class containing 5 through the site 4. Second, note that it takes an even number of steps to get from 1 to 4. Third, note that it takes an odd number of steps to get from 4 to 5. Therefore, we can only get from 1 to 5 in an odd number of steps. So $P\{X_n = 5 | X_0 = 1\} = 0$ if n is even. If, on the other hand, n is odd, then to get to 5 we first need to get to its communicating class, which from our computation in (f.2) has a probability of $1/7$ (the entry on the first row and first column, since we are going from 1 to R). Once we are in the appropriate communicating class, we know we are starting at 4 (which is specific to this Markov chain!) and we are asking about the probability of being at 5 in an appropriate number of steps. This is given by the invariant measure (of that class) at 5 times the period. So if we n is going along even numbers, then $P\{X_n = 5|X_0 = 1\}$ converges to $1/7 \times \Phi_\infty(5) \times 2 = 1/7 \times 1/4 \times 2 = 1/14$.

Similarly, starting at 1, $P\{X_n = 7|X_0 = 1\} = 0$ if n is even and converges to $1/7 \times \Phi_\infty(7) \times 2 = 1/7 \times 1/4 \times 2 = 1/14$ if n is going to infinity along the odd numbers. On the other hand, $P\{X_n = 4|X_0 = 1\} = 0$ if n is odd (because we need an even number of steps to get to 4 from 1) and converges to $1/7 \times \Phi_\infty(4) \times 2 = 1/7 \times 1/8 \times 2 = 1/28$ if $n \to \infty$ along even numbers. And $P\{X_n = 6|X_0 = 1\} = 0$ if n is odd (because we need an even number of steps to get to 4 from 1 and an even number of steps to get from 4 to 6) and converges to $1/7 \times \Phi_\infty(6) \times 2 = 1/7 \times 3/8 \times 2 = 3/28$ as $n \to \infty$ along even numbers. Observe how adding over all the possibilities gives $0 + 1/14 + 0 + 1/14 = 1/7$ along the odd n and $1/28 + 0 + 3/28 + 0 = 1/7$ along the even n. This $1/7$ is the probability (as $n \to \infty$) we are caught in one of the states $\{4, 5, 6, 7\}$, starting at 1, which is what we computed in (f). If you think about this a little longer, you will understand exactly why it was necessary to multiply by the period!