MATH 5040/6810: Homework 2 (Due Monday, September 23)

Problem 1
  Let N be an integer with $N \geq 4$. Consider a Markov chain with state space $\{1, \ldots, N\}$. From 1 the chain moves to 3 or N equally likely. From a site j between 2 and $N-2$ the chain goes to $j-1$ or $j+2$ equally likely. From $N-1$ the chain jumps to 1 or $N-2$ equally likely and from N it goes to 2 or $N-1$ equally likely. Can you guess an invariant measure for the Markov chain? Explain your guess and then prove it is correct. Hint: Arrange the states 1 through N on a circle and draw the graphical representation of the Markov chain. What do you notice about the sites 1 through N?

Problem 2
  Consider the Markov chain on state space $\{a, b, c\}$. From a the chain goes to b with probability $1/3$ and to c with probability $2/3$. From b it goes to a or c equally likely. From c it goes to straight to a. Is this chain irreducible? What is its period? Compute the invariant measure by hand. Do not use a computer or a calculator for this problem.

Note: If you do not learn how to do this computation by hand on such a small example, you may struggle during exam(s)!

Problem 3
  Consider the Markov chain on state space $\{4, 5, 6, 7\}$. From 4 the chain goes to 5 or 7 equally likely. From 5 it goes to 4 or 6 equally likely. From 6 it goes to 5 or 7 equally likely. From 7 it goes straight to 6. Is this chain irreducible? What is its period? Compute the invariant measure.

Problem 4
  Consider the Markov chain on state space $\{2\}$. From 2 it goes back to 2. What is the invariant measure for this chain?

Problem 5
  Consider the Markov chain on state space $\{1, 2, 3, 4, 5, 6, 7, 8\}$. From 1 it goes to 2 or 3 equally likely. From 2 it goes back to 2. From 3 it goes to 1, 2, 4, or 8 equally likely. From 4 the chain goes to 5 or 7 equally likely. From 5 it goes to 4 or 6 equally likely. From 6 it goes to 5 or 7 equally likely. From 7 it goes straight to 6. From 8 the chain goes back to 8.

(a) What are the communicating classes? Which are recurrent and which are transient? What are their periods?

(b) Since the transition matrix is stochastic we know it has an eigenvalue equal 1. What is the multiplicity of this eigenvalue?

Note: It is NOT a coincidence that multiplicity of 1 matches the number of recurrent communicating classes.

(c) Use your answer to Problems 3 and 4 to give three different invariant measures for this Markov chain.

(d) Prove that if you add a third of each of the invariant measures you found in (c) you will get again an invariant measure. What about half of one, a quarter of another, and a quarter of the third? How many invariant measures does this Markov chain have?!

(e) If you start the Markov chain at 1, what is the expected number of returns to 1? Compute this in two ways:

   (e.1) By observing that from 1 you can go to 2, you can go to 3 then leave to 2, 4, or 8, or you can go to 3 then return to 1. With the first four moves you will never return to 1. Reduce this problem to a much simpler Markov chain (with only two states!) and find the mass function of the number of total times your return to 1, i.e. the probability it equals 0, 1, 2, etc. Now compute the average number of returns to 1.

   (e.2) Use the linear algebra method you learned in class. In this case, compute also the average number of visits to 3, starting at 1. Starting at 1, what is the average number of steps it takes before you leave the transient class forever?

(f) Starting at 1, what is the probability you will ever reach 2? Again, compute this in two ways:

   (f.1) Let p be the probability we are after. Now observe that starting at 1 you can go to 2 directly, go to 3 then to 4 or 8 and thus never reach 2, go to 3 and then to 2, or go to 3 and back to 1 and now the story starts over again. Use this observation to write and solve a simple equation for p.

   (f.2) Use the linear algebra ideas you learned in class.

(g) Starting at 1, what is the probability you will ever reach 8? (You could solve this in two ways too, but this time pick your favorite way of doing it.)

(h) Starting at 4, what is the expected time of first return to 4?

(i) Starting at 4, what is the expected number of steps before you reach 5?

(j) What is the limit of $P\{X_n = 5 | X_0 = 1\}$ as $n \to \infty$?

Hint: You have computed the probability the chain ends up absorbed at 2 and at 8. This also gives you the probability the chain ends up in the communicating class containing 5. Also, in this particular chain, you know exactly where the chain enters that class. Once the chain is in that class and you give it a long time it reaches the equilibrium of that class, which you computed in part (c). Now answer the question. Be careful about the period, though.