MATH 5040/6810: HOMEWORK 1 (DUE MON, SEP. 9)

Problem 1

a) Let $A_0, A_1, A_2, \ldots$ be events such $\mathbb{P}(A_0 \cap \ldots \cap A_n) > 0$ for all integers $n \geq 0$. Show that for all integers $n \geq 1$

$$\mathbb{P}(A_0 \cap \cdots \cap A_n) = \mathbb{P}(A_0)\mathbb{P}(A_1|A_0) \cdots \mathbb{P}(A_n|A_0 \cap \cdots \cap A_{n-1}).$$

b) Consider a stochastic process $\{X_0, X_1, \ldots\}$ on a state space $S = \{a, b, c\}$. What does the equation in part a) say about the probability $\mathbb{P}\{X_0 = a, X_1 = b, X_2 = c, X_3 = b, X_4 = c\}$?

c) How does the equation you got in part b) simplify if we assume the process in part b) to be a Markov process?

Problem 2

Consider the following Markov chain on state space $\{a, b, c\}$: the chain starts at a, b, or c with probability, respectively, 1/2, 1/6, and 1/3. Then, from a the chain either stays at a with probability 1/3, or goes to c; from b the chain goes to a or c with probability 1/2 each; and from c the chain goes to a, b, or c, with probabilities 1/2, 1/4, and 1/4, respectively.

a) Write the transition matrix of the Markov chain.

b) Using the formula you developed in 1c) compute the probability $\mathbb{P}\{X_0 = a, X_1 = c, X_2 = b, X_3 = c, X_4 = b\}$.

c) Compute the probability $\mathbb{P}\{X_1 = c\}$ by considering all the possible scenarios leading to the event $X_1 = c$.

d) Compute $\mathbb{P}\{X_0 = a | X_1 = c\}$.

e) Compute the probability $\mathbb{P}\{X_5 = b\}$. (Hint: do NOT consider all possible scenarios leading to the event $X_5 = b$! It is OK to use a calculator to compute powers of matrices.)

f) Compute the invariant measure of the Markov chain. (OK to use a computer or a calculator.)

g) Compute the limit of $\mathbb{P}\{X_n = b\}$ as $n \to \infty$. (Hint: Invariant measure.)

Problem 3

You have a bag with 5 red balls and 3 blue ones. You also have an infinite reserve of red and blue balls. Every minute you draw a ball at random from the bag and look at its color. Then you return it to the bag and add with it, to the bag, another ball of that same color (which you take from the infinite reserve). Let X_n denote the color of the ball you drew at time n. Is X_n a Markov chain? Why or why not? If it is, write its transition matrix.

Problem 4

You have \$4 in your pocket. You are given a coin that lands heads with probability p. Every minute you toss the coin once. If it is heads you gain an extra \$1. If it is tails you lose a \$1. If you reach \$0 then you will have \$0 for the rest of the game, regardless of what your coin comes up. The (small) House has \$6. So if you reach \$10 you will have \$10 for the rest of the game regardless of what your coin lands.

Is this a Markov chain? Why or why not? If it is, write its transition matrix.

Problem 5

Consider a tiny web consisting of 6 sites. Site 1 is a John's homepage. John teaches calculus and his site points to sites 2 and 3. Site 2 is the course syllabus and does not point to any other sites. Site 3 is the calculus course website. It points back at John's homepage and also at the course syllabus. It also points at Emily's webpage, site 5. (Emily is the TA for the course.) Site 4 belongs to a friend of Emily's, Jack. It points at Emily's website and at Jack's old website, site 6. Emily's website points at both Jack's pages, the new one 4 and the old one 6. Jack's old webpage 6 points at his new webpage 4.

a) Draw the graph corresponding to the above network.

b) Consider the stochastic process where from each site you go equally likely to any of the sites it points to. What happens when you eventually reach site 2?

c) Extend the stochastic process by the rule: from a site that does not point at any other site go equally likely to any site on the web (including the original site itself). Does this process give a Markov chain? Write its transition matrix. What are its communication classes? Which are transient and which are recurrent? Compute the invariant measure. Does it give positive probability to all sites? Can we use it to rank the transient states? Why or why not? (It is OK to use a computer or a laptop to compute the probability measure.)

d) To overcome the problem with transient states, consider instead the following Markov chain: from a site that does not point to any other site, go equally likely to any site on the web (including the site itself); but from a site that does points somewhere, flip a coin that gives heads with probability 15%. If the coin lands heads, go equally likely to any site on the web (including the site itself). If, on the other hand, it lands tails then go equally likely to any site that the current site points to. Write the transition matrix for this chain. What are its communication classes? Which are transient and which are recurrent? Compute the invariant measure. Does it give positive probability to all sites?

e) Use your results in part d) to rank the six sites.