Chapter 18

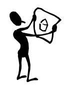
Two-Sample Problems

Two-Sample Problems

- The goal of inference is to compare the responses to two treatments or to compare the characteristics of two populations.
- We have a separate sample from each treatment or each population.
  - Each sample is separate. The units are not matched, and the samples can be of differing sizes.

Case Study

Exercise and Pulse Rates



A study if performed to compare the mean resting pulse rate of adult subjects who regularly exercise to the mean resting pulse rate of those who do not regularly exercise.

	n	mean	std. dev.
Exercisers	29	66	8.6
Nonexercisers	31	75	9.0

This is an example of when to use the two-sample t procedures.

Conditions for Comparing Two Means

- We have **two <u>independent</u> SRSs**, from two distinct populations
  - that is, one sample has no influence on the other--matching violates independence
  - we measure the same variable for both samples.
- Both populations are **Normally distributed**
  - the means and standard deviations of the populations are unknown
  - in practice, it is enough that the distributions have similar shapes and that the data have no strong outliers.

Two-Sample *t* Procedures

- In order to perform inference on the difference of two means ($\mu_1 - \mu_2$), we'll need the standard deviation of the observed difference $\bar{x}_1 - \bar{x}_2$:

$$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Two-Sample *t* Procedures

- ***Problem:*** We don't know the population standard deviations σ_1 and σ_2.
- ***Solution:*** Estimate them with s_1 and s_2. The result is called the <u>standard error</u>, or estimated standard deviation, <u>of the difference</u> in the sample means.

$$SE = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Two-Sample t Confidence Interval

- Draw an SRS of size n_1 form a Normal population with unknown mean μ_1, and draw an independent SRS of size n_2 form another Normal population with unknown mean μ_2.
- A **confidence interval for $\mu_1 - \mu_2$** is:

$$\left(\bar{x}_1 - \bar{x}_2\right) \pm t^* \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

  – here t^* is the critical value for confidence level C for the t density curve. The **degrees of freedom** are equal to the smaller of $n_1 - 1$ and $n_2 - 1$.

Case Study
Exercise and Pulse Rates

Find a 95% confidence interval for the difference in population means (nonexercisers minus exercisers).

$$\bar{x}_1 - \bar{x}_2 \pm t^* \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = 75 - 66 \pm 2.048 \sqrt{\frac{(9.0)^2}{31} + \frac{(8.6)^2}{29}}$$
$$= 9 \pm 4.65$$
$$= 4.35 \text{ to } 13.65$$

"We are 95% confident that the difference in mean resting pulse rates (nonexercisers minus exercisers) is between 4.35 and 13.65 beats per minute."

Two-Sample t Significance Tests

- Draw an SRS of size n_1 form a Normal population with unknown mean μ_1, and draw an independent SRS of size n_2 form another Normal population with unknown mean μ_2.
- To **test the hypothesis H_0: $\mu_1 = \mu_2$**, the test statistic is:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- Use P-values for the t density curve. The degrees of freedom are equal to the smaller of $n_1 - 1$ and $n_2 - 1$.

P-value for Testing Two Means

- H_a: $\mu_1 > \mu_2$
  - P-value is the probability of getting a value as large or larger than the observed test statistic (t) value.
- H_a: $\mu_1 < \mu_2$
  - P-value is the probability of getting a value as small or smaller than the observed test statistic (t) value.
- H_a: $\mu_1 \neq \mu_2$
  - P-value is *two times* the probability of getting a value as large or larger than the absolute value of the observed test statistic (t) value.

Case Study
Exercise and Pulse Rates

Is the mean resting pulse rate of adult subjects who regularly exercise different from the mean resting pulse rate of those who do not regularly exercise?

- Null: The mean resting pulse rate of adult subjects who regularly exercise is the *same* as the mean resting pulse rate of those who do not regularly exercise? [H_0: $\mu_1 = \mu_2$]
- Alt: The mean resting pulse rate of adult subjects who regularly exercise is *different* from the mean resting pulse rate of those who do not regularly exercise? [H_a : $\mu_1 \neq \mu_2$]

Degrees of freedom = 28 (smaller of 31 – 1 and 29 – 1).

Case Study

1. ***Hypotheses:*** H_0: $\mu_1 = \mu_2$ H_a: $\mu_1 \neq \mu_2$
2. ***Test Statistic:***

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{75 - 66}{\sqrt{\frac{(9.0)^2}{31} + \frac{(8.6)^2}{29}}} \approx 3.961$$

3. ***P-value:***
   P-value = $2P(T > 3.961) = 0.000207$ *(using a computer)*
   P-value is smaller than 2(0.0005) = 0.0010 since $t = 3.961$ is greater than $t^* = 3.674$ (upper tail area = 0.0005) *(Table C)*
4. ***Conclusion:***
   Since the P-value is smaller than $\alpha = 0.001$, there is very strong evidence that the mean resting pulse rates are different for the two populations (nonexercisers and exercisers).

Robustness of *t* Procedures

- The two-sample *t* procedures are more robust than the one-sample *t* methods, particularly when the distributions are not symmetric.
- When the two populations have similar distribution shapes, the probability values from the *t* table are quite accurate, even when the sample sizes are as small as $n_1 = n_2 = 5$.
- When the two populations have different distribution shapes, larger samples are needed.
- In planning a two-sample study, it is best to choose equal sample sizes. In this case, the probability values are most accurate.

Using the *t* Procedures

- Except in the case of small samples, the assumption that each sample is an independent SRS from the population of interest is more important than the assumption that the two population distributions are Normal.
- ***Small sample sizes ($n_1 + n_2 < 15$):*** Use *t* procedures if each data set appears close to Normal (symmetric, single peak, no outliers). If a data set is skewed or if outliers are present, do not use *t*.
- ***Medium sample sizes ($n_1 + n_2 \geq 15$):*** The *t* procedures can be used except in the presence of outliers or strong skewness in a data set.
- ***Large samples:*** The *t* procedures can be used even for clearly skewed distributions when the sample sizes are large, roughly $n_1 + n_2 \geq 40$.

Details of *t* Degrees of Freedom

- Using degrees of freedom as the smallest of $n_1 - 1$ and $n_2 - 1$ is only a rough approximation to the actual degrees of freedom for the two-sample *t* procedures.
- A better approximation that is used by software uses a function of the sample sizes and sample standard deviations to compute degrees of freedom *df*.
- Use of *df* from the software calculation gives more accurate results than when simply using the smaller of $n_1 - 1$ and $n_2 - 1$.

Details of *t* Degrees of Freedom

APPROXIMATE DISTRIBUTION OF THE TWO-SAMPLE *t* STATISTIC

The distribution of the two-sample *t* statistic is very close to the *t* distribution with degrees of freedom df given by

$$\text{df} = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1 - 1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{s_2^2}{n_2}\right)^2}$$

This approximation is accurate when both sample sizes n_1 and n_2 are 5 or larger.

Case Study

Exercise and Pulse Rates

Compute the degrees of freedom *df* used by software to analyze these data using two-sample *t* procedures.

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1 - 1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{s_2^2}{n_2}\right)^2} = \frac{\left(\frac{(9.0)^2}{31} + \frac{(8.6)^2}{29}\right)^2}{\frac{1}{30}\left(\frac{(9.0)^2}{31}\right)^2 + \frac{1}{28}\left(\frac{(8.6)^2}{29}\right)^2} \approx 57.97$$

This is the degrees of freedom value used by software when computing critical values and *P*-values.

Avoid Inference About Standard Deviations

- There are methods for inference about the standard deviations of Normal populations.
- Most software packages have methods for comparing the standard deviations.
- However, these methods are extremely sensitive to non-Normal distributions and this lack of robustness does not improve in large samples.
- Hence it is not recommended that one do inference about population standard deviations in basic statistical practice.