$\boxed{\text{K3 surfaces}}$ (AG Student seminar)

<u>Def</u>: X sm. proj surface$/\mathbb{C}$ is called a K3 surface if $K_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.
(or $K_X = 0$)

<u>Rmk</u>: $K_X = 0 \Rightarrow \kappa(X) = 0$ and K3's are actually characterized as the only sfcs w/ $\kappa(X) = 0$, $q = 0$ and $p_g = 1$.
(Class. of surfaces)

<u>NB</u>: We defined K3's to be projective. One can define an <u>analytic</u> K3 to be a complex manifold w/ $\dim X = 2$, X connected & compact and the same requests on cohomology.
By GAGA, the above def. is a special case of the analytic one. <u>Result by Siu</u>: analytic K3's are Kähler. Not all K3's are projective (I'll state some examples).

$\dim(\text{Moduli of analytic K3's}) = 20$
$\dim(\text{Moduli of algebr. K3's}) = 19$ (divisor)

Betti & Hodge #'s computation

X connected. $h^{0,0}(X) = 1 = h^{2,2}(X)$ by Serre duality.

$q = h^{0,1} = h^{1,0} = h^{2,1} = h^{1,2} = 0$.

K_X trivial $\Rightarrow$ $1 = h^{2,0}$. So also $h^{0,2} = 1$.

Result:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & ? & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Before finding ? a remark: $h^{1,0} = 0$ has a geometric interpretation. Recall $K_X = 0$, so the natural pairing

$$\Omega^1_X \times \Omega^1_X \longrightarrow \Omega^2_X \cong \mathcal{O}_X \quad \text{induces an isom:}$$

$$T_X = \Omega_X^* = \mathcal{H}om(\Omega_X, \mathcal{O}_X) \cong \Omega_X.$$

$H^0(X, \Omega_X) = 0$ then also means $H^0(X, T_X) = 0$, no global vector fields. (K3's are very "pointy").

Finding $h^{1,1}$: $\chi(X, \mathcal{O}_X) = \sum (-1)^i h^i(\mathcal{O}_X) = 2$ from previous computations.

By Noether's formula (special case of HRR) we get

$$\chi(\mathcal{O}_X) = \frac{c_1(X)^2 + c_2(X)}{12}. \quad K_X = 0 \Rightarrow c_1(X) = 0 \leftarrow \text{degree of } \det T_X.$$

Because $\Lambda^2 T_X = \Lambda^2 \Omega_X$ by above

So $2 = \dfrac{c_2(X)}{12} \Rightarrow c_2(X) = 24$

But $c_2(X) = \chi_{top} = \sum (-1)^i b_i \Rightarrow b_2 = 22$ and $? = 20$.

TELL LATER

Other immediate consequences of the definition:

$K_X = 0 \Rightarrow$ RR becomes $\chi(L) = \underset{\substack{\uparrow \\ \text{by above}}}{2} + \frac{(L.L)}{2}$.

So $(L.L)$ always even!

Moreover if $L = \mathcal{O}_X(C)$ for $C \subseteq X$ curve, the genus formula gives $(C.C) = 2g-2$.

(since $g = h^1(\mathcal{O}_C)$ and $h^0(\mathcal{O}_C) = 1$ $\uparrow$ flavor of "canonical curves"

We'll talk about this later

Now: <u>EXAMPLES</u> (Too often left untreated)

(1) <u>Quartics in $\mathbb{P}^3$.</u> Indeed from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0$$

Since accidentally $\mathcal{O}_{\mathbb{P}^3}(-4)$ is the canonical, we get by Serre duality $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$

This forces $H^1(X, \mathcal{O}_X) = 0$.

Again using the accident and adjunction we get

$$K_X = (K_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4))\big|_X = \mathcal{O}_X. \quad \checkmark$$

$\{x^4 + y^4 + z^4 + w^4 = 0\}$ "Fermat quartic".

② Complete intersections $\subseteq \mathbb{P}^N$

let $S = S_{d_1,\dots,d_n}$ complete inters. of n hypersurf. in $\mathbb{P}^{n+2}$.
To avoid trivialities, $d_i \geq 2$ (otherwise we are just looking at $\mathbb{P}^N$ for some smaller N).

- $H^1(S, \mathcal{O}_S) = 0$ since S is a complete intersection (induction on dimension using exact seq. or use Lefschetz theorem on hyperplane sections)

- For a complete intersection, Hartshorne teaches:
$K_S = \mathcal{O}_S\left(\sum_{i=1}^n d_i - n - 3\right)$. Imposing $\sum d_i - n - 3 = 0$ yields 3 cases: S_4, $S_{2,3}$, $S_{2,2,2}$.

③ (Usually neglected example) Double plane.

$\pi : X \longrightarrow \mathbb{P}^2$ double cover branched over a sextic.
C = zero locus of $s \in H^0(\mathcal{O}_{\mathbb{P}^3}(6))$. Look at $\mathcal{O}(6)$ as $\mathcal{O}(3)^{\otimes 2}$.
"Square root" e.b.
In general for $\pi : X \xrightarrow{d:1} Y$ w/ branching over a divisor w/ $\mathcal{O}(C) \simeq \mathcal{L}^{\otimes d}$ we get
$\pi_* \mathcal{O}_X \simeq \mathcal{O}_Y \otimes \mathcal{L}^\vee \otimes \dots \otimes (\mathcal{L}^\vee)^{d-1}$. $\mathcal{L} = \mathcal{O}(3)$ for us
Therefore for us $\pi_* \mathcal{O}_X \simeq \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)$. So $H^1(\pi_* \mathcal{O}_X) = 0$ by finiteness of the mapping $H^1(X, \mathcal{O}_X) = 0$.
As for K_X, general theory of branched coverings again gives $K_X \simeq \pi^*(\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{L}^{d-1}) = \pi^* \mathcal{O}_{\mathbb{P}^3} = \mathcal{O}_X$.
here we need sextic & deg = 2.

4 Kummer K3's.

A ab-sfc. Involution σ on A. Consider the 16 fixed points $\{p_i\}$

$$\begin{tikzcd} \varepsilon : \tilde{A} \arrow[r, "Bl"] \arrow[d, "\pi"'] & A \arrow[d] \\ \tilde{A}/\pm\tilde{\sigma} = X \arrow[r, "resol."] & A/\{\pm 1\} \end{tikzcd}$$

($\cancel{16}$ 16 sing. pts)

$\tilde{\sigma}$ on $\tilde{A}$ is the id on the exceptionals and -1 elsewhere.

- X smooth (obvious if you believe X resolves $A/\{\pm 1\}$) just check that local coord. descend to X from the exceptionals on $\tilde{A}$.

(x,y) coord near p_i. $\sigma^* x = -x$ $\sigma^* y = -y$.
$x' := \varepsilon^* x$ $y' := \varepsilon^* y$.

$(x', y'/x' = t)$ coord near $q \in E_i = \varepsilon^{-1}(p_i)$. Since $\tilde{\sigma}^* x' = -x'$ but $\tilde{\sigma}^* t = t$, we have that $u = (x')^2$ and t are coordinates around q, and they descend to X.

- $K_X = 0$. A has a hol. 2-form $\omega = dx \wedge dy$ locally. $\sigma^* \omega = \omega$ so $\varepsilon^* \omega$ is invariant under $\tilde{\sigma}^*$.

therefore it descends to an α on X, s.t. $\varepsilon^* \omega = \pi^* \alpha$.

Using the above notation

$$\pi^* \alpha = \varepsilon^* \omega = dx' \wedge dy' = dx' \wedge d(tx') = x' dx' \wedge dt = \tfrac{1}{2} du \wedge dt$$

which is non vanishing near q so α does not vanish and $K_X = 0$. (K_X is the divisor of *any* α!)

- $H^1(X,\mathcal{O}_X)=0$. $H^1(X,\mathcal{O}_X)\neq 0$ would imply that $\tilde{A}$ has
$\overset{\shortparallel}{H^0(X,\Omega^1)}$ an invariant $(1,0)$ form under $\tilde{\sigma}$.
By birat. of the blow up, also A would have one. but $H^0(A,\Omega^1_A)$ has $\{dx, dy\}$ as a basis and they are not invariant.

<u>Rmk</u>: Non-projective K3's come up here.
If A is an ab. surface, then K3 is projective.
But if we do this w/ A complex torus which is not projective, then X turns out to be a non-projective K3.

- <u>Next</u>: - "genus g" K3's and canonical map.
- lattice for H^2 and Torelli theorems.

Suppose we have $C \subseteq X$ smooth of genus g.

~~Def A K3 w/ a smooth curve of genus g on it is called "a genus g" K3.~~

~~Important result: genus g K3's exist $\forall g \geq 3$~~

→ <u>Consequences of having $C \subseteq X$ of genus g</u>

$(C, C) = 2g-2$ ✓

$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ gives $h^1(\mathcal{O}_X(-C)) = 0$

So by Serre also $h^1(\mathcal{O}_X(C)) = 0$.

Since $K_X = 0$.

R-R $\Rightarrow$ $h^0(X, \mathcal{O}_X(C)) = g+1$, i.e. $\dim |C| = g$.

$\varphi_C : X \overset{|C|}{\dashrightarrow} \mathbb{P}^g$. Actually this has no base points, since it can only have them on C, but when restricted to C $|C|$ cuts $|K_C|$

(we get this by adjunction: $K_C = \mathcal{O}_X(C)|_C$ by $K_X = 0$)

K_C is known to be bpf from the theory of curves.

So we have $\varphi_C : X \longrightarrow \mathbb{P}^g$ $\forall g \geq 1$.

Moreover $C \hookrightarrow \mathbb{P}^{g-1} \subseteq \mathbb{P}^g$ via the usual canonical morphism under φ_C, as a hyperpl. section of $\varphi_C(X)$.

Properties of φ_C ($g \geq 2$, $g=1$ contracts the curve)
some problem.

Prop: Let $C \subseteq X$ smooth, $g \geq 2$.

① • If $g=2$, then $\varphi_C : X \longrightarrow \mathbb{P}^2$ is the double plane

② • If $g \geq 3$ either φ_C is birational and the generic $D \in |C|$ is non hyperelliptic

or φ_C is 2:1 to a rational surface of degree $g-1$ in $\mathbb{P}^g$, and the generic $D \in |C|$ is hyperelliptic.

proof: ① $g=2 \Rightarrow C$ hyperelliptic. So $\varphi_C|_C$ has deg 2.

$\varphi^{-1}\varphi(C) = C$

$(C.C)=2$ and $\mathcal{O}_X(C) = \varphi^*(\mathcal{O}(1))$ so actually φ_C has degree 2 everywhere.

To find the degree of B = branch locus, look at $\varphi(C) = \ell \subseteq \mathbb{P}^2$ a line. $C \xrightarrow{2:1} \ell$, branched over d points $= B \cap \ell$. Using $g=2$ and Hurwitz we get $d=6$.

② Assume $D \in |C|$ general non hyperell. Then $\varphi|_D$ is an embedding. $\varphi^{-1}\varphi(C) = C$ implies then that φ is birational, since a general point $x \in X$ will lie on a general $D \in |C|$.

If φ is not birational, then $D \in |C|$ is hyperelliptic and φ has degree 2.
Since $(C,C) = 2g-2$ $\varphi(X)$ has degree $g-1$ ($\varphi(C)$ has deg $g-1$, and $\varphi(C) = \varphi(X) \cap H$)
Since $D \in |C|$ sweeps X and the D's are hyperell, then the $\varphi(D)$'s are lines that sweep $\varphi(X)$ (i.e. $\varphi(X)$ is rational).

Rmk: one usually looks at polarized K3's
i.e. a K3 + a primitive ample line bundle
~~of degree~~ w/ pairing $(L.L) = 2d$.
One says the K3 has degree d.
If $L = \mathcal{O}(C)$, then $2d = 2g-2$ and the K3 has "genus g".

Analysis of H^2 and Torelli

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

all the fun is in the H^2.

$H^2(X,\mathbb{Z}) \cong E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$ abstractly.

signature of the lattice $= (3, 19)$

This comes from Hodge Index theorem

If we restrict to $H^{1,1}$ we have $(1, 19)$ and

$NS \hookrightarrow H^{1,1}$ has $(1, \rho(X)-1)$.

Def. X K3. A marking is an isomorphism $\varphi : H^2(X,\mathbb{Z}) \cong \Lambda$

~~Torelli theorem: $X \cong X'$ if and only if [illegible] (X', L') give the same marking~~

Torelli's theorem: $(X, L) \cong (X', L')$ if and only if $\exists$ Hodge isometry $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ taking ℓ to ℓ' ($[L]=\ell$, $[L']=\ell'$)

(equivalent to period mapping being injective).