TORIC VARIETIES

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Abstract. This is a very short introduction to some concepts around toric varieties, some of the subsections are intended for more experienced algebraic geometers. To see a lot of exercises and get more involved with fan structures you can read [2], and to learn more about Cox rings you can read [1].

1. Affine Toric Varieties

1.1. **Affine Toric Varieties from characters.** In this notes, we denote by $\mathbb{K}$ an algebraically closed field of characteristic zero. We denote by N a free finitely generated abelian group and by M its dual. Given an element $m = (a_1, \dots, a_n) \in M$, this gives a character $\chi^m \colon (\mathbb{K}^*)^n \to \mathbb{K}^*$, defined by

$$\chi^m(t_1, \dots, t_n) = t_1^{a_1} \cdots t_n^{a_n}.$$

Observe that any character of $\mathbb{K}^*$ arises in this way. Given a finite set $\mathcal{A} = \{m_1, \dots, m_s\} \subset M$, this gives a map

$$\Phi_{\mathcal{A}} \colon T_N \to \mathbb{K}^s,$$

where $T_N = \mathrm{Spec}(\mathbb{K}[M]) \simeq (\mathbb{K}^*)^n$, where n denotes the rank of N. The *affine toric variety* $Y_{\mathcal{A}}$ is the Zariski closure of $\Phi_{\mathcal{A}}(T_N)$. The dimension of Y_α is the rank $\mathbb{Z}\mathcal{A}$.

Given the map $\mathbb{Z}^s \to M$ mapping the canonical basis $e_1, \dots, e_s$ to $m_1, \dots, m_s$, we denote by L the kernel inducing a short exact sequence

$$0 \to L \to \mathbb{Z}^s \to M,$$

given $l \in L$ we set

$$l_+ = \sum_{l_i > 0} l_i e_i \text{ and } l_- = -\sum_{l_i < 0} l_i e_i,$$

where $l = (l_1, \dots, l_s) \in L$. The ideal $I_{\mathcal{A}}$ defining the affine toric variety $Y_{\mathcal{A}} \subset \mathbb{C}^s$ is

$$\langle x^{l_+} - x^{l_-} \mid l \in L \rangle.$$

In particular, the ideal defining an affine toric variety is always binomial.

1.2. **Affine Toric Varieties via cones.** Let $\sigma \subset N_{\mathbb{Q}}$ be a polyhedral pointed cone, then its dual $\sigma^\vee \subset M_{\mathbb{Q}}$ induces a semigroup $S_\sigma = \sigma^\vee \cap M$, therefore we have a induced ring $\mathbb{K}[S_\sigma]$, so we obtain an affine toric variety

$$U_\sigma = \mathrm{Spec}(\mathbb{K}[S_\sigma]) = \mathrm{Spec}(\mathbb{K}[\sigma^\vee \cap M]),$$

is an affine toric variety. If σ is full-dimensional and pointed, then $\sigma^\vee$ is full-dimensional and pointed as well, the maximal ideal of $\mathbb{K}[S_\sigma]$ induced by the full-dimensional face of $\sigma^v ee$ corresponds to the unique fixed point of U_σ by the torus action.

We say that a semigroup $S \subset M$ is *saturated* if for each $k \in \mathbb{N} - \{0\}$ and $m \in M$, $km \in S$, implies that $m \in S$. For example, if $\sigma \subset N_{\mathbb{R}}$ is strongly convex rational polyedral cone, then $S_\sigma = \sigma^\vee \cap M$ is saturated.

Theorem 1.1. *Let V be an affine toric variety with torus T_N. Then the following are equivalent:*

- *V is a normal algebraic variety,*
- *$V = \operatorname{Spec}(\mathbb{K}[S])$, where $S \subset M$ is saturated affine semigroup,*
- *$V = \operatorname{Spec}(\mathbb{K}[S_\sigma]) \simeq U_\sigma$, where $S_\sigma = \sigma^\vee \cap M$ and $\sigma \subset N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone.*

More generally, there is a correspondence between the category of strongly convex rational polyhedral cones with linear morphism and affine normal toric varieties with T_N-equivariant morphisms. Moreover, there is a one-to-one dimension-reversing bijection between the subcones of σ and the oribits of U_σ. Clearly, since every ray of σ is saturated, the singularities of U_σ appears in codimension 2.

1.3. **The Normalization Morphism.** Given an affine toric variety V, its corresponds to the spectrum of the ring over a semigroup $S \subset M$, meaning that $V \simeq \mathbb{K}[S]$, let $\operatorname{Cone}(S)$ the cone generated over the positive rational numbers by the elements of S, then we can set $\sigma = \operatorname{Cone}(S)^\vee \subset N_{\mathbb{R}}$, this will be a rational polyedral cone, and we can associate a toric variety U_σ. The inclusion of rings $\mathbb{K}[S] \subset \mathbb{K}[\sigma^\vee \cap M]$ induces a birational morphism of toric varieties $U_\sigma \to V$ that is the normalization map of V. In other words, the normalization process for toric varieties corresponds to the saturation of the corresponding semigroup.

1.4. **Smoothness of Affine Toric Varieties.** We say that a cone $\sigma \subset N_{\mathbb{Q}}$ is *smooth* if $\sigma \cap N$ has exactly $\operatorname{rank}(N)$ extremal rays and it spans N over $\mathbb{Z}$. We say that a ccone $\sigma \subset N_{\mathbb{Q}}$ is simplicial if it is full-dimensional and has exactly $\operatorname{rank}(N)$ extremal rays. Since automorphisms of $N_{\mathbb{Q}}$ induces isomorphic toric varieties we can see that any smooth cone $\sigma \subset N_{\mathbb{Q}}$ induces a smooth affine toric variety U_σ and any simplicial cone $\sigma \subset N_{\mathbb{Q}}$ induces a finite quotient singularity affine toric variety U_σ, which is still a $\mathbb{Q}$-manifold.

2. Projective Toric Varieties

Consider the following exact sequence of tori

$$1 \to \mathbb{K}^* \to (\mathbb{K}^*)^{n+1} \to T_{\mathbb{P}^n} \to 1,$$

where $T_{\mathbb{P}^n}$ represents the big torus of the projective space $\mathbb{P}^n$. We denote the right hand morphism of the above exact sequence by π. Consider a set $\mathcal{A} = \{m_1, \dots m_s\} \subset M$ as before, and compose $\Phi_{\mathcal{A}}$ with π, and we define the projective toric variety $X_{\mathcal{A}}$ to be the Zariski closure in $\mathbb{P}^{s-1}$ of the map $\pi \colon \Phi_{\mathcal{A}}$. $X_{\mathcal{A}}$ is a toric variety of dimension equal to the dimension of the smallest affine subspace containing $\mathcal{A}$, and moreover $Y_{\mathcal{A}}$ is the affine cone of $X_{\mathcal{A}}$.

2.1. **projective varieties via polytopes.** Given a polytope $P \subset M_{\mathbb{R}}$ we will say that it is *normal* if

$$(kP) \cap M + (lP) \cap M = ((k+l)P) \cap M,$$

for all $k, l \in \mathbb{N}$, where $+$ denotes the Minkowski sum. Observe that $P \subset M_{\mathbb{R}}$ is a full-dimensional lattice polytope of dimension $n \geq 2$. Then kP will be normal whenever

$k \geq n-1$. As a corollary we conclude that any polytope in $\mathbb{R}^2$ is normal. We say that a lattice polytope $P \subset M_{\mathbb{R}}$ is *very ample* if for every vertex $m \in P$, the semigroup $S_{P,m} = \mathbb{N}(P \cap M - m)$ generated by the set $P \cap M - m = \{m' - m \mid m' \in P \cap M\}$ is saturated in M. Observe that a normal polytope P is always very ample.

Let $P \subset M_{\mathbb{R}}$ be a very ample polytope relative to the lattice M, and let $\dim(P) = n$. If $P \cap M = \{m_1, \ldots, m_s\}$, then $X_{P\cap M}$ is the Zariski closure of the image of the map $T_N \to \mathbb{P}^{s-1}$ given by $t \mapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)) \in \mathbb{P}^{s-1}$. If we fix homogeneous coordinates $x_1, \ldots, x_s$ for $\mathbb{P}^{s-1}$, we can see that for each $m_i \in P \cap M$ the semigroup

$$S_i = \mathbb{N}(P \cap M - m_i),$$

induces the affine toric variety $X_{P\cap M} \cap U_i$, where U_i is the open affine set of $\mathbb{P}^{s-1}$ corresponding to $x_i \neq 0$. Meaning that we have an isomorphism of schemes (if we forget about the schematic structure, this is just an isomorphism of the underrlying analytic variety).

$$X_{P\cap M} \cap U_i \simeq \mathrm{Spec}(\mathbb{K}[S_i]),$$

and therefore we have that

$$X_{P\cap M} = \bigcup_{m_i \text{ vertex of } P} X_{P\cap M} \cap U_i.$$

2.2. **Fan structure of a complete toric variety.** We define a *fan* Σ to be a finite set of pointed convex polyhedral cones of $N_{\mathbb{Q}}$ such that the face of any cone in Σ is again in Σ, and the intersection of two cones in Σ is a face of both. Given a fan Σ we can define a toric variety, denoted by $X(\Sigma)$, by gluing the affine toric varieties $X(\sigma)$ and $X(\sigma')$ along $X(\sigma \cap \sigma')$ whenever σ and σ' are cones of Σ. Any μ-dimensional face of σ defines a $(n-\mu)$-dimensional orbit of $X(\sigma)$. The above gluing is along T-invariant subvarieties, then $X(\Sigma)$ is also endowed with a T-action. We denote by $\Sigma(\mu)$ the set of μ-dimensional faces of Σ. Therefore, given a fan Σ, the T_N-invariant prime divisors of $X(\Sigma)$ are in one-to-one correspondence with the rays of Σ, and then we can ask whenever a linear combination of such divisors D_ρ is an ample or very ample divisor of the corresponding toric variety. We will see later that any polytope that is normal, and in particular very ample, comes with a natural ample divisor in our toric variety.

2.3. **From polytopes to Projective toric varieties.** Now, we recall the construction of projective toric varieties from polytopes. Given a full-dimensional convex compact polytope $P \subset M_{\mathbb{Q}}$, we denote by Σ_P its dual fan and $X(P)$ the toric variety associated to the dual fan. Observe that we can write

$$P = \{m \in M_{\mathbb{Q}} \mid \langle m, \rho_i \rangle \leq -d_i, \text{ for } i \in \{1, \ldots, r\}\}$$

for certain integers $d_1, \ldots, d_r$. The divisor $D_P = \sum_{i=1}^r d_i D_i$ defines an ample divisor on $X(P)$, and we say that D_P is the divisor of $X(P)$ associated to the polytope P. Observe that different polytopes can define exactly the same toric variety if they have the same dual fan, but the associated divisors will define different embeddings into projective spaces. We say that a polytope is smooth if it defines a smooth toric variety.

2.4. **Fan and polytope duality.** Given a polytope, we can constrcut a normal dual fan of the polytope, such that both, the fan and the polytope induce the same toric variety. The idea, is that we can embedd our polytope in $\Delta \subset N_{\mathbb{Q}}$, and the we look at the function $\min\langle \Delta, - \rangle \colon M_{\mathbb{Q}} \to \mathbb{Q}$, and this function will be piecewise linear, we define the cones of our fan by the cones where this function is linear.

3. Cox Construction

3.1. **The Cox Ring.** In what follows, we will denote by $X(\sigma)$ the affine toric variety induced by the rational pointed polyhedral cone $\sigma \in N_{\mathbb{Q}}$.

Given an algebraic variety X, we denote by $\mathrm{WDiv}(X)$ the group of Weil divisors of X and $\mathrm{PDiv}(X)$ the group of principal divisors of X. For a toric variety $X(\Sigma)$, we denote by $\mathrm{WDiv}_T(X(\Sigma))$ and $\mathrm{PDiv}_T(X(\Sigma))$ the group of T-invariant Weil divisors and T-invariant principal divisors, respectively. There is a bijection between the T-invariant divisors of $X(\Sigma)$ and the one-dimensional faces of Σ. We denote by $\rho_1, \dots, \rho_r$ the primitive lattice generators of the one-dimensional faces of Σ, and by D_i the T-invariant divisor associated to ρ_i for each i.

Given an algebraic variety X which is irreducible, normal, with only constant invertible functions and finitely generated divisor class group, we define its Cox rings to be

$$\mathcal{R}(X) = \bigoplus_{[D] \in \mathrm{Cl}(X)} H^0\left(X, \mathcal{O}_X(D)\right).$$

3.2. **Construction of the Cox Ring.** In what follows we will fix a canonical basis $\{e_1, \dots, e_n\}$ of N, given an element $\rho \in N_{\mathbb{Q}}$ we denote by ρ_i its i-th coordinate on this basis. In what follows, to make the computation easier, we will assume that the cone $\sigma = \langle -e_1, \dots, -e_n \rangle$ is always a cone of the fan of our torus. We will compute the Cox ring for this kind of toric varieties (this means, that the projective toric variety that we are looking, contains an invariant point that is smooth).

Let $\rho_1, \dots, \rho_r$ denote the primitive generators of the one-dimensional faces of Σ, with $r \geq n+1$ and $\rho_i = -e_i$ for $1 \leq i \leq n$. Let $F = \mathbb{Z}^r$ and consider the linear map $P \colon F \to N$ sending the i-th canonical base vector $f_i \in F$ to $\rho_i \in N$. Denote by $\delta \subset F_{\mathbb{Q}}$ the positive orthant and define a fan $\widehat{\Sigma}$ in F consisting on all the faces of δ whose image on N are contained in some cone of Σ. Then P induces a toric morphism

and we have exact sequences

$$0 \longrightarrow M \xrightarrow{P^*} E \xrightarrow{Q} K \longrightarrow 0 \tag{3.1}$$

$$0 \longrightarrow K^* \xrightarrow{Q^*} F \xrightarrow{P} N \longrightarrow 0,$$

where E is the dual of F, $P^* \colon M \to E$ the dual map of P and $Q \colon E \to K = E/P^*(M)$ the induced projection. Since $\rho_i = -e_i$ for $1 \leq i \leq n$, we can write

$$P = \begin{bmatrix} -\mathrm{Id}_n & P_0 \end{bmatrix} \qquad Q = \begin{bmatrix} P_0^t & \mathrm{Id}_{r-n} \end{bmatrix},$$

where P_0 is the $n \times (r-n)$ matrix whose columns are the vectos $\rho_{n+1}, \dots, \rho_r$. Observe that Q induces a K-grading on $\mathbb{K}[E \cap \delta^\vee]$, then we conclude that

$$\mathrm{Cox}(X(\Sigma)) \simeq \mathbb{K}[E \cap \delta^\vee]$$

as K-graded polynomial rings.

Recall that the group $\mathrm{WDiv}^T(X(\Sigma))$ of T-invariant Weil divisors of $X(\Sigma)$ is generated by the T-invariant divisors D_i corresponding to the rays ρ_i of Σ. Then, we have an isomorphism $E \simeq \mathrm{WDiv}^T(X(\Sigma))$ given by

$$e \mapsto \langle e, f_1\rangle D_1 + \cdots + \langle e, f_r\rangle D_r.$$

Moreover, the injective morphism P^* identifies M with $\mathrm{PDiv}^T(X(\Sigma))$. Thus, we conclude that $K \simeq \mathrm{Cl}(X(\Sigma))$ and we can write

$$\mathrm{Cox}(X(\Sigma)) \simeq k[x_1, \dots, x_r], \tag{3.2}$$

$$\deg(x_i) = [D_i] \in \mathrm{Cl}(X(\Sigma)), \quad \mathrm{Spec}(\mathrm{Cox}(X(\Sigma))) \simeq \mathbb{K}^r.$$

Denoting by $H = \mathrm{Spec}(\mathbb{K}[K])$ the torus acting on $\mathrm{Cox}(X(\Sigma))$ we have that the subvariety $X(\widehat{\Sigma}) \subset \mathrm{Spec}(\mathrm{Cox}(X(\Sigma)))$ is H-invariant and the morphism p is a good quotient for the induced action on $X(\widehat{\Sigma})$. In the coordinates 3.2 the complement $V(\Sigma)$ of $X(\widehat{\Sigma})$ in $\mathbb{K}^r$ has defining ideal

$$\mathrm{Irr}(\Sigma) = \left\langle \prod_{i\in I} x_i \mid I \subset \{1, \dots, r\} \text{ and } \{\rho_i \mid i \in I\} \text{ are not the rays of a cone of } \Sigma \right\rangle,$$

called the *irrelevant ideal* of $\mathrm{Cox}(X(\Sigma))$.

3.3. **Automorphism group of a Complete Toric Variety.** Now we turn to describe the automorphisms of $X(\Sigma)$, which are induced by H-equivariant automorphisms of $X(\widehat{\Sigma})$. First, observe that any element $t \in T$ defines an automorphism of X so we can identify $T \subset \mathrm{Aut}(X(\Sigma))$. We denote by $\mathrm{Aut}(N, \Sigma)$ the subgroup of automorphism of N preserving the fan Σ, any such automorphism induces an automorphism of $X(\Sigma)$. Finally, we say that $m \in M$ is a *Demazure root* of Σ if the following condition holds:

$$\text{There exists } i \in \{1, \dots, r\} \text{ such that } \langle m, \rho_i\rangle = -1 \text{ and } \langle m, \rho_j\rangle \geq 0, \text{ for all } j \neq i.$$

We also say that such m is a *Demazure root of* ρ_i. Observe that given a Demazure root m of ρ_i and $t \in \mathbb{K}^*$ we have a K-graded automorphism of $\mathrm{Cox}(X(\Sigma))$ defined by

$$y_{(m,t)}(x_i) = x_i + t \prod_{j\neq i} x_j^{\langle m, \rho_j\rangle} \text{ and } y_{(m,t)}(x_j) = x_j \text{ for all } j \neq i. \tag{3.3}$$

This automorphism induces an automorphism of $X(\Sigma)$. We denote by $\mathcal{R}(\Sigma)$ the set of automorphisms induced by Demazure roots on $X(\Sigma)$. By abuse of notation we also denote by $\mathcal{R}(\Sigma)$ the set of K-graded automorphisms of $\mathrm{Cox}(X(\Sigma))$ defined by 3.3. With the above notation, we state the following theorem proved by David Cox.

Theorem 3.1. *Let $X(\Sigma)$ be a complete simplicial toric variety. Then $\mathrm{Aut}(X(\Sigma))$ is generated by $T, \mathcal{R}(\Sigma)$ and $\mathrm{Aut}(N, \Sigma)$.*

References

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[2] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR2810322 (2012g:14094) ↑1

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