MATH 5620 – NUMERICAL ANALYSIS II
PRACTICE MIDTERM EXAM

Problem 1. Consider the multistep method:

$$y_n - y_{n-2} = \frac{h}{3}[f_n - 3f_{n-1} + 2f_{n-2}]$$

(a) Is this method implicit or explicit?
(b) Is this method convergent, stable and/or consistent? Justify your answer.

Problem 2.
(a) Write pseudocode for the Jacobi method for solving a linear system $\mathbf{Ax} = \mathbf{b}$.
(b) Show that if $\mathbf{A}$ is diagonally dominant then $\|\mathbf{I} - \mathbf{D}^{-1}\mathbf{A}\|_\infty < 1$, where $\mathbf{D} = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$. Recall that a matrix $\mathbf{A}$ is diagonally dominant when

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \; i = 1, \ldots, n.$$

(c) Show that if $\mathbf{A}$ is diagonally dominant then the Jacobi iteration converges.
(d) Is diagonally dominance necessary for convergence of the Jacobi iteration?

Problem 3.
(a) Using the method of undetermined coefficients derive the second order Adams-Moulton formula of the form

$$y_{n+1} = y_n + h[Af_{n+1} + Bf_n]$$

(b) Recall that for a linear multistep method of the form

$$a_k y_n + a_{k-1} y_{n-1} + \cdots + a_0 y_{n-k} = h\left[b_k f_n + b_{k-1} f_{n-1} + \cdots + b_0 f_{n-k}\right]$$

we associate the linear functional

$$Ly = \sum_{i=0}^{k} [a_i y(ih) - h b_i y'(ih)]$$

which has Taylor expansion at $t = 0$:

$$Ly = d_0 y(0) + d_1 y'(0) + d_2 h^2 y''(0) + \cdots$$

where

$$d_0 = \sum_{i=0}^{k} a_k$$

$$d_j = \sum_{i=0}^{k} \left(\frac{i^j}{j!} a_i - \frac{i^{j-1}}{(j-1)!} b_i \right), \; j \geq 1.$$

Verify that this is an order 2 method and find the local truncation error, i.e. find the constant C for which

$$y(t_n) - y_n = C h^3 y^{(3)}(t_{n-1}) + \mathcal{O}(h^4),$$

where the previous values y_{n-1}, y_{n-2}, $\ldots$ are assumed exact.

Problem 4. Consider the IVP

$$\begin{cases} y' = f(t,y) \\ y(a) = \alpha. \end{cases} \tag{1}$$

(a) Write the first 3 terms of the Taylor series for $y(t+h)$ expanding around t. Your series should be in terms of h, f and its partial derivatives. The residual should be $\mathcal{O}(h^3)$.

(b) Recall the general formula for a second-order Runge-Kutta method:

$$y(t+h) = y + w_1 hf + w_2 hf(t+\alpha h, y+\beta hf) + \mathcal{O}(h^3), \tag{2}$$

where $y \equiv y(t)$ and $f \equiv f(t,y)$. Use the two variable Taylor expansion

$$f(t+hs, y+hv) = f(t,y) + hsf_t(t,y) + hvf_y(t,y) + \mathcal{O}(h^2)$$

to express (2) in terms of y, f, $f_t \equiv f_t(t,y)$ and $f_y \equiv f_y(t,y)$.

(c) What conditions should w_1, w_2, α and β satisfy in order for the method to be second order?

(d) Write down the particular Runge-Kutta method of order 2 with $w_1 = 1/4$.