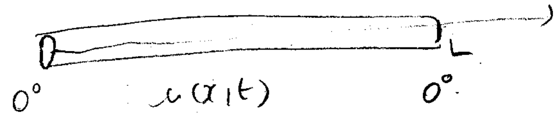
§ 3.5 The One Dim. Heat eq (1DHEQ)



$u(x,t)$ = temperature distribution on a rod of length L, with ice bath on both ends.

$u(x,t)$ satisfies Heat Equation:

$$\begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0^\circ & \text{B.C. = ice bath} \\ u(x,0) = f(x) & \text{I.C. = initial temperature distrib.} \end{cases}$$

Use method of separation of variable (it helps to see what we did last time in § 3.3)

Ansatz $u(x,t) = X(x)T(t)$

$$XT' = c^2 X'' T \Rightarrow \underbrace{\frac{T'}{c^2 T}}_{F(t)} = \underbrace{\frac{X''}{X}}_{G(x)} \underset{\text{must}}{=} k = \text{const indep of } x \text{ and } t.$$

We get 2 equations:

$$\begin{cases} X'' - kX = 0 \\ X(0) = X(L) = 0 \leftarrow \text{from B.C.} \end{cases}$$

$k = -\mu^2$, $\mu_n = \frac{n\pi}{L}$

$$\boxed{X_n(x) = \sin \frac{n\pi}{L} x} \quad n = 1, 2, \ldots$$

and $T' - kc^2 T = 0$

$$T_n' + \left(\frac{cn\pi}{L}\right)^2 T_n = 0$$

$$\boxed{T_n(t) = b_n \exp\left[-\left(\frac{cn\pi}{L}\right)^2 t\right]}$$

$n = 1, 2, \ldots$

Thus we get a fundamental mode:

$$u_n(x,t) = b_n \sin\left[\frac{n\pi}{L} x\right] \exp\left[-\left(\frac{cn\pi}{L}\right)^2 t\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right) \exp\left[-\left(\frac{cn\pi}{L}\right)^2 t\right]$$

1DHEQ is homogeneous & lin.

Now what about initial conditions?

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad \boxed{= \text{Sine Series of } f}$$

$$\Rightarrow b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

exp. decay of temp

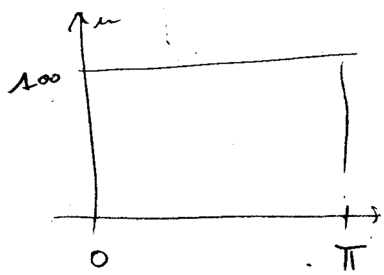
Summary

$$\begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

is solved by $u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \exp\left[-\left(\frac{cn\pi}{L}\right)^2 t\right]$

where $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$.

Example



$$\begin{cases} u_t = u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = 100 \end{cases}$$

$$\rightsquigarrow u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \exp[-n^2 t]$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} 100 \sin nx \, dx$

$$= -\frac{200}{\pi n} \cos nx \Big|_{x=0}^{\pi} = \frac{200}{n\pi}(1-(-1)^n)$$

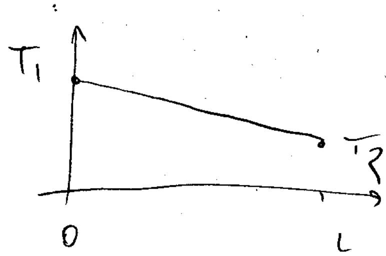
$$u(x,t) = \sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t}}{2k+1} \sin(2k+1)x$$

$\boxed{\text{Show matlab code}}$

Steady state temp distrib:

$$u_t = 0 \Rightarrow u_{xx} = 0$$

$$\Rightarrow u(x) = Ax + B$$



$$\boxed{u(x)} = T_1 \frac{L-x}{L} + T_2 \frac{x-0}{L}$$

$$= \boxed{\frac{T_2 - T_1}{L} x + T_1}$$

Other Boundary Conditions:

$$(A)\begin{cases} u_t = c^2 u_{xx} \\ u(0,t) = T_1 \\ u(L,t) = T_2 \\ u(x,0) = f(x) \end{cases}$$

① find steady state

$$s(x) = \frac{T_2 - T_1}{L} x + T_1$$

② shift solution by steady state

let $\boxed{v = u - s}$ then

$$v_t = u_t$$

$$v_{xx} = u_{xx}$$

$\Rightarrow$ v solves:

$$(B)\begin{cases} v_t = c^2 v_{xx} \\ v(0,t) = 0 \\ v(L,t) = 0 \\ v(x,0) = f(x) - s(x) \end{cases}$$

And we know how to solve (B):

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin \frac{n\pi}{L} x$$

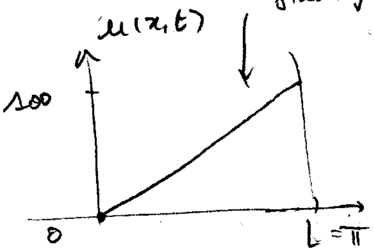
where $b_n = \frac{2}{L}\int_0^L (f(x) - s(x)) \sin \frac{n\pi}{L} x \, dx$

③ go back to original problem

$$\boxed{u(x,t) = v(x,t) + s(x)}$$

Example:

steady state temp $s(x) = \frac{100}{\pi} x$



$\Rightarrow$ I.C. for (B) is $100 - \frac{100}{\pi} x$

solve (B) for:

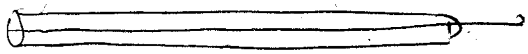
$$b_n = \frac{2}{\pi}\int_0^{\pi} \left(100 - \frac{100}{\pi} x\right) \sin nx \, dx$$

$$= \frac{200}{n\pi}$$

$$\Rightarrow \boxed{u(x,t) = s(x) + \sum_{n=1}^{\infty} b_n \sin nx \; e^{-n^2 t}}$$

$$= \frac{100}{\pi} x + \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{\sin nx}{n} e^{-n^2 t}$$

§3.6 Even more boundary conditions for 1D Heat Eq



0 L

metal rod with insulated ends

homogeneous Neumann B. C.

$$(1)\begin{cases} u_t = c^2 u_{xx}, & 0<x<L, \ t>0 \\ u_x(0,t) = u_x(L,t) = 0, & t>0 \quad \Leftrightarrow \text{ no heat flux} \\ u(x,0) = f(x) & 0<x<L \end{cases}$$

Solution using separation of variable.

$u(x,t) = X(x)T(t)$ = ansatz, plugging in (1) gives:

$$\begin{cases} X'' - kX = 0 \\ X'(0) = X'(L) = 0 \end{cases} \qquad T' - kc^2 T = 0$$

$k = \mu^2 > 0$: $X(x) = a\cosh\mu x + b\sinh\mu x$ + B.C. $\Rightarrow$ $X(x) = 0$ (Check)

$k = 0$: $X(x) = ax + b$ + B.C. : $X'(x) = a$

$X'(0) = X'(L) = 0 \Rightarrow a = 0$

$\Rightarrow$ we get $X(x) = b$ a nontrivial solution

$k = -\mu^2 < 0$: $X(x) = a\cos\mu x + b\sin\mu x$

$X'(x) = -a\mu\sin\mu x + b\mu\cos\mu x$

$X'(0) = 0 \Rightarrow b = 0$

$X'(L) = 0 \Rightarrow a\sin\mu L = 0$

$\Rightarrow \mu = \mu_n = \frac{n\pi}{L}$, $n = 1, 2, \dots$

$\rightsquigarrow X_n(x) = \cos\frac{n\pi}{L}x$, $n = 1, 2, \dots$

Thus we get:

$T_0(t) = a_0$

$T_n(t) = a_n \exp\left[-\left(\frac{n\pi c}{L}\right)^2 t\right]$

Using superposition principle:

$$\boxed{u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left[-\left(\frac{n\pi c}{L}\right)^2 t\right] \cos\left(\frac{n\pi}{L} x\right)}$$

solves by construction (1)?

What about initial conditions?

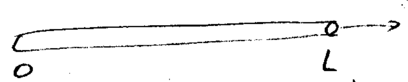
$$u(x,0) = f(x) = a_0 + \sum_{n=1} a_n \cos\left(\frac{n\pi}{L} x\right)$$

surprise! a_n, $n \geq 0$ are the coeff in the <u>cosine series</u> of $f(x)$.

$$\Rightarrow \quad a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi}{L} x \, dx$$

Now more complicated Robin type boundary cdts:



heat transfer / convection } coefficient

Dirichlet
$u(0,t)=0$
(ice bath)

Robin
$u_x(L,t) = -\mathcal{K} u(L,t)$

Bar loses heat at a rate proportional to temperature of medium.

$$(2) \begin{cases} u_t = c^2 u_{xx}, & 0<x<L, \ t>0 \\ u_x(0,t) = 0, & t>0 \\ u_x(L,t) = -\mathcal{K} u(L,t), & t>0 \\ u(x,0) = f(x), & 0<x<L \end{cases}$$

<u>Idea</u> Use separation of variables to get:

$$\begin{cases} X'' - kX = 0 \\ X(0) = 0 & \text{(BC1)} \\ X'(L) = -\mathcal{K} X(L) & \text{(BC2)} \end{cases} \qquad T' - kc^2 T = 0$$

<u>$k = \mu^2 > 0$</u>: $X(x) = a\cosh \mu x + b \sinh \mu x$. (BC1) $\Rightarrow$ $a=0$

(BC2) $\Rightarrow$ $b \underbrace{\sinh \mu L}_{>0} = 0 \Rightarrow b=0$

$\Rightarrow X(x) = 0$ (trivial sol)

<u>$k=0$</u> $X(x) = ax+b$. (BC1) $\Rightarrow$ $b=0$

(BC2) $\Rightarrow$ $a = -\mathcal{K} a L \Rightarrow a(1+\mathcal{K}L) = 0$

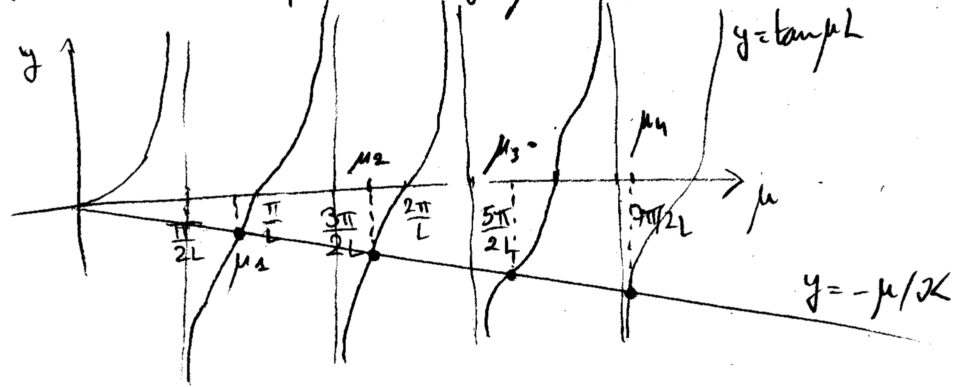
$\Rightarrow a = 0$. (trivial sol)

$k = -\mu^2$: $X(x) = a\cos\mu x + b\sin\mu x$

(BC1) $\Rightarrow$ $a = 0$

(BC2) $\Rightarrow$ $\mu\cos\mu L = -K\sin\mu L \iff \tan\mu L = -\dfrac{\mu}{K}$

gives a non linear eq to solve for μ:



can solve (numerically) for roots μ_n, $n \geq 1$

$\leadsto$ get solutions: $X_n(x) = \sin\mu_n x$

$T_n(t) = c_n \exp\left[-c^2\mu_n^2 t\right]$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \exp\left[-c^2\mu_n^2 t\right] \sin\mu_n x}$$ solves (2) by construction

what about I.C.?

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\mu_n x$$ $\sim$ Sine series but not quite "generalized Fourier series"

It turns out (Sturm-Liouville theory, §6) that

$\{\sin\mu_n x\}_{n=1}^{\infty}$ form an orthogonal system of functions

w.r.t inner product: $(u,v) = \int_0^L u(x)\,v(x)\,dx$.

thus

$$(f, \sin\mu_m x) = \sum_{n=1}^{\infty} c_n (\sin\mu_n x, \sin\mu_m x)$$

$$= c_m (\sin\mu_m x, \sin\mu_m x)$$

$$\Rightarrow c_m = \frac{(f, \sin\mu_m x)}{(\sin\mu_m x, \sin\mu_m x)}$$