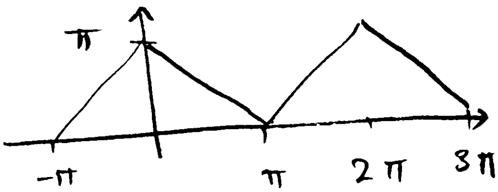
If $f(x)$ is continuous, we have no Gibbs phenomenon in Fourier series of $f(x)$.

Example: Triangular wave



$$g(x) = \begin{cases} \pi - x & \text{if } 0 \le x \le \pi \\ \pi + x & \text{if } -\pi \le x \le 0 \end{cases}$$

and 2π - periodic.

Fourier coeff are:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi} \text{ area of triangle w/ base} = 2\pi,\ \text{height} = \pi$$

$$= \frac{1}{2\pi}\left(2\pi \times \pi \times \frac{1}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cos nx}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} \underset{\downarrow}{(\pi - x)}\ \underset{\uparrow}{\cos nx}\,dx$$

$$\overset{\text{IBP}}{=} \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\sin nx}{n}\,dx\right].$$

$$= \frac{2}{\pi}\left(-\frac{\cos nx}{n^2}\right)\Big|_0^{\pi} = \frac{2}{\pi n^2}\left(1 - \cos n\pi\right)$$

$$= \frac{2}{\pi n^2}\left(1 - (-1)^n\right) = \begin{cases} \frac{4}{\pi n^2} & \text{for } n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{\text{odd}}\,dx = 0$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left(1 - (-1)^n\right)\cos nx \quad + \text{pi/2}$$

keeping only NZ terms.

$$= \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)^2}\cos\left((2k+1)x\right) + \text{pi/2}$$

Show Falstad
(No Gibbs)

Linear combination of Fourier series

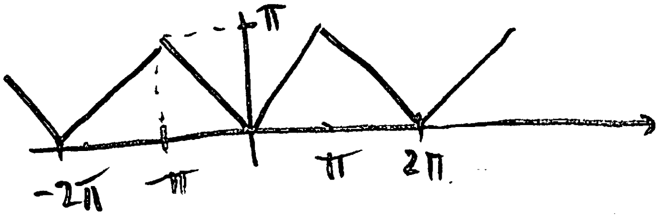
If:
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$
$$g(x) = \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos nx + \tilde{b}_n \sin nx$$

Then:
$$(f+g)(x) = (a_0+\tilde{a}_0) + \sum_{n=1}^{\infty} (a_n+\tilde{a}_n)\cos nx + (b_n+\tilde{b}_n)\sin nx$$

(see example 2.2.4 in text book)

Change of Variables:

example: compute Fourier series of translated triangular wave



$h(x) = |x|$, for $-\pi \le x \le \pi$.

Note: $h(x) = g(x+\pi)$ use F.S. of $g(x)$ eval at $x+\pi$.

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}(1-(-1)^n)\cos(n(x+\pi))$$
$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left(1-(-1)^n\right)(-1)^n \cos nx$$

Fourier coeff of $h(x)$ are: $a_0 = \frac{\pi}{2}$, $a_n = \frac{2}{\pi n^2}((-1)^n-1)$

Scratch paper: $\cos(n(x+\pi)) = \cos nx \underbrace{\cos n\pi}_{(-1)^n} - \underbrace{\sin nx \sin n\pi}_{=0}$

§2-3 Fourier series of functions w/ arbitrary period

Change of variables: $y = \frac{\pi}{p} x$

Inner prod	$\int_{-p}^{p} u(x)v(x)dx$	$\int_{-\pi}^{\pi} f(y)\,g(y)dy$
period:	$2p$	2π
	$f(x)$ $g(\frac{\pi}{p}x)$	$f(\frac{p}{\pi}y)$ $g(y)$
orthogonal family	$1 \quad \cos\frac{\pi}{p}x \dots \cos\frac{n\pi p}{x} \dots$ $\sin\frac{\pi}{p}x \dots \sin\frac{n\pi p}{x} \dots$	$1 \quad \cos \cancel{x}\,y \dots \cos n\cancel{x}\,y \dots$ $\sin \cancel{x}\,y \dots \sin n\cancel{x}\,y \dots$

should be
n*pi*x/p

Proof: see HW2, which can be done using COV:

$(1,1) = 2p$

$$\left(\cos\frac{n\pi}{p}x, \cos\frac{m\pi x}{p}\right) = \int_{-p}^{p} \cos\frac{n\pi x}{p}\cos\frac{m\pi x}{p}\,dx$$

$$\overset{\text{COV } x=\frac{p}{\pi}y}{=} \frac{p}{\pi}\int_{-\pi}^{\pi} \cos ny \sin my\, dy$$

$$= \frac{p}{\pi}\begin{cases} \pi & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$

$$= \begin{cases} p & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$

$(1,1) = 2\pi$

$$(\cos nx, \cos mx) = \begin{cases} \pi & \text{if } n=m \\ 0 & n\neq m \end{cases}$$

$$(\sin nx, \sin mx) = \begin{cases} \pi & \text{if } n=m \\ 0 & n\neq m \end{cases}$$

$$(\sin nx, \cos mx) = 0$$

$\perp$ conditions are: for interval $[-p, p]$:

$$(1, 1) = 2p$$

$$\left(\cos\frac{n\pi}{p}x, \cos\frac{m\pi}{p}x\right) = \begin{cases} p & \text{if } n=m \\ 0 & n \neq m \end{cases}$$ (see HW2)

$$\left(\sin\frac{n\pi}{p}x, \sin\frac{m\pi}{p}x\right) = \begin{cases} p & \text{if } n=m \\ 0 & n \neq m \end{cases}$$

$$\left(\sin\frac{n\pi}{p}x, \cos\frac{m\pi}{p}x\right) = 0$$

Theorem Let f be a $2p$ periodic pcws smooth function. It's Fourier series expansion is:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x$$

where:

$$a_0 = \frac{(f, 1)}{(1, 1)} = \frac{1}{2p}\int_{-p}^{p} f(x)\,dx$$

$$a_n = \frac{\left(f, \cos\frac{n\pi}{p}x\right)}{\left(\cos\frac{n\pi}{p}x, \cos\frac{n\pi}{p}x\right)} = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{\left(f, \sin\frac{n\pi}{p}x\right)}{\left(\sin\frac{n\pi}{p}x, \sin\frac{n\pi}{p}x\right)} = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

The series converges pointwise to $\dfrac{f(x+) + f(x-)}{2}$