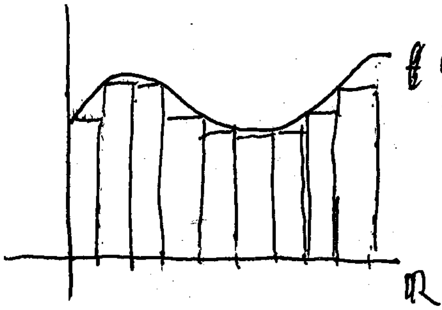
Riemann - Stieltjes Integration



Def: Let $[a,b]$ be a given interval, a *partition* P of $[a,b]$ is a finite set of points

$$a = t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n = b$$

we write $\Delta t_i = t_i - t_{i-1}$

We define, corresponding to a partition P, the *upper and lower Riemann sums*

which are upper/lower bounds on the area under the graph of f, $f : [a,b] \to \mathbb{R}$ bounded

$$U(P,f) = \sum_{i=1}^{n} M_i (t_i - t_{i-1}) \qquad M_i = \sup_{t_{i-1} \leq t \leq t_i} f(t)$$

$$L(P,f) = \sum_{i=1}^{n} m_i (t_i - t_{i-1}) \qquad m_i = \inf_{t_{i-1} \leq t \leq t_i} f(t)$$

The upper and lower Riemann Integrals are defined

$$\overline{\int_a^b} f\,dx = \inf U(P,f)$$

$$\underline{\int_a^b} f\,dx = \sup L(P,f)$$

geometrically clear

$\underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx$

If these integrals give the same value

then we say f is <u>Riemann-Integrable</u>

and write

$\int_a^b f\,dx$ for the value $\inf U(P,f)$
$= \sup L(P,f)$

Since f bdd $\exists$ m, M w/

$$m \leq f \leq M \quad \text{on } [a,b]$$

so that

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

this $\Rightarrow$ upper and lower integrals are well defined

We will also consider some more general ~~integration~~ integrals with will allow us to give (one type of) meaning to integration against δ-fcns

$$\int f(t)\,\delta_0(t)\,dt = f(0)$$

Let $\alpha : [a,b] \to \mathbb{R}$ be monotone increasing and $f:[a,b] \to \mathbb{R}$ bdd

define upper and lower sums

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i (\alpha_{t_i} - \alpha_{t_{i-1}})$$
$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i (\alpha_{t_i} - \alpha_{t_{i-1}})$$

then

$$\overline{\int_a^b} f(t)\,d\alpha = \inf_P U(P, f, \alpha)$$
$$\underline{\int_a^b} f(t)\,d\alpha = \sup_P L(P, f, \alpha)$$

upper and lower Riemann-Stieltjes integrals

and if their values agree we name it

$\int_a^b f(t)\,d\alpha$ the Riemann-Stieltjes Integral

Note if $\alpha(t) = t$ we get original Riemann integral.

If α is C^1 (we will see) that

$$\int_a^b f \, d\alpha = \int_a^b f \, \alpha'(t) \, dt \quad \text{(usually)}$$

but for general monotone α we could get something more interesting.

If $\int_a^b f \, d\alpha$ exists we say $f \in \mathcal{R}(\alpha)$.

We want to understand which functions are Riemann integrable and/or in $\mathcal{R}(\alpha)$ for some weight $d\alpha$.

We need to understand what happens when we change the partition

Def: We say that a partition P_* is a refinement of P if $P^* \supset P$. Given two partitions P_1 and P_2 we say that $P_* = P_1 \cup P_2$ is their <u>common refinement</u>

Thm If P_* is a refinement of P then

$$L(P, f, \alpha) \leq L(P_*, f, \alpha)$$
$$U(P, f, \alpha) \geq U(P_*, f, \alpha)$$

proof: We suppose that P_* just has one more point than P. The general case follows by adding one point at a time.

Call t_* the extra point

$$t_{i-1} < t_* < t_i \quad \text{for some } i$$

$$L(P_*, f, \alpha) - L(P, f, \alpha) = \inf_{[t_{i-1}, t_*]} f\,(\alpha_{t_*} - \alpha_{t_{i-1}}) + \inf_{[t_*, t_i]} f\,(\alpha_{t_i} - \alpha_{t_*})$$
$$- \inf_{[t_{i-1}, t_i]} f\,(\alpha_{t_i} - \alpha_{t_{i-1}})$$

Then using α monotone and

$$\inf_{[t_{i-1}, t_*]} f \,,\ \inf_{[t_*, t_i]} f \geq \inf_{[t_{i-1}, t_i]} f = m_i$$

$$L(P_*, f, \alpha) - L(P, f, \alpha) \geq m_i \left(\alpha_{t_i} - \alpha_{t_*} + \alpha_{t_*} - \alpha_{t_{i-1}} - (\alpha_{t_i} - \alpha_{t_{i-1}})\right)$$
$$= 0$$

$\square$

Thm $\underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha$

proof: let $P_* = P_1 \cup P_2$ the common refinement of P_1, P_2 then

$$L(P_1, f, \alpha) \leq L(P_*, f, \alpha) \leq U(P_*, f, \alpha) \leq U(P_2, f, \alpha)$$

taking inf over P_2
then sup over P_1
yields the result. $\square$

Thm $f \in \mathcal{R}(\alpha)$ on $[a,b]$ iff $\forall$ $\varepsilon > 0$ $\exists$ a partition P w/

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

proof: Suppose for all $\varepsilon > 0$ $\exists$ such a partition P_1 then

$$\overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\, d\alpha \le U(P_1, f, \alpha) - L(P_1, f, \alpha) < \varepsilon$$

$\varepsilon > 0$ arbitrary $\Rightarrow$ $\overline{\int} f\, d\alpha = \underline{\int} f\, d\alpha$

Suppose $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha = \int_a^b f\, d\alpha$ Riemann Integral exists

Then $\forall$ $\varepsilon > 0$ $\exists$ P_1 s.t.

$$\int_a^b f\, d\alpha < L(P_1, f, \alpha) + \varepsilon/2$$

P_2 s.t.

$$\int_a^b f\, d\alpha > U(P_2, f, \alpha) - \varepsilon/2$$

Let $P = P_1 \cup P_2$ the common refinement

then same ineq hold for P so

$$U(P, f, \alpha) - L(P, f, \alpha) < \int_a^b f\, d\alpha + \varepsilon/2 - \int_a^b f\, d\alpha + \varepsilon/2 = \varepsilon$$

$\blacksquare$

Thm (a) if $\left(U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon \right)$ (*)

then same holds for any refinement of P

(b) If (*) holds for P and ~~s_i, t_i~~

$x_i, y_i \in [t_{i-1}, t_i]$ are arbitrarily chosen points of each sub-interval of P then

$$\sum_{i=1}^{n} |f(x_i) - f(y_i)| (\alpha_{t_i} - \alpha_{t_{i-1}}) < \varepsilon$$

(i.e. $$\left| \sum_{i=1}^{n} f(x_i)(\alpha_{t_i} - \alpha_{t_{i-1}}) - \underline{\int_a^b} f(t)\,dt \right| < \varepsilon$$

or $\overline{\int_a^b} f(t)\,dt$)

(c) If $f \in \mathcal{R}(\alpha)$ and (*) holds then

$$\left| \sum_{i=1}^{n} f(x_i)(\alpha_{t_i} - \alpha_{t_{i-1}}) - \int_a^b f\,dt \right| < \varepsilon$$

(i.e. any other choice of Riemann sum wrt P approx just as well)

proof. easy

f is cts on $[a,b]$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

Pf: f cts on $[a,b]$ (cpct) $\Rightarrow$ unif cts. Let $\varepsilon > 0$

$\exists$ $\delta > 0$ s.t. $|t-s| < \delta$ $\Rightarrow$

$(\alpha_b - \alpha_a) \cdot |f(t) - f(s)| < \varepsilon$ for $t, s \in [a,b]$
$|t-s| < \delta$

Let P be any partition w/

$$\text{mesh}(P) = \sup_i |t_i - t_{i-1}| < \delta$$

then

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)(\alpha_{t_i} - \alpha_{t_{i-1}})$$

$$\leq \frac{\varepsilon}{\alpha_b - \alpha_a} \sum_{i=1}^{n} \alpha_{t_i} - \alpha_{t_{i-1}}$$

(telescoping sum) $\leq \varepsilon$

ε arbitrary $\Rightarrow$ $f \in \mathcal{R}(\alpha)$.

~~Now we look more carefully at this proof and~~
~~try to extract more out of it.~~

~~Let us first~~

Thm If f monotonic on $[a,b]$ and α is cts on $[a,b]$ then $f \in \mathcal{R}(\alpha)$.

Remark: In a sense we are integrating by parts $\int_a^b f d\alpha = (f\alpha)_a^b - \int_a^b \alpha \, df$

since f is a legitimate weight and $\alpha \in \mathcal{R}(f)$ since α cts.

proof: $$\Big[U(P,f,\alpha) = \sum_{i=1}^{n} M_i(\alpha_{t_i} - \alpha_{t_{i-1}})$$

(monotonicity) $$= \sum_{i=1}^{n} f(t_i)(\alpha_{t_i} - \alpha_{t_{i-1}})$$

$$= \sum_{i=1}^{n} f(t_i)\alpha_{t_i} - \sum_{i=0}^{n-1} f(t_{i+1})\alpha_{t_i}$$

$$= f(t_n)\alpha_{t_n} - f(t_1)\alpha_{t_0} - \sum_{i=1}^{n-1} \alpha_{t_i}(f(t_{i+1}) - f(t_i))$$

$$= f(b)\alpha_b - f(t_1)\alpha_a - \sum_{i=1}^{n-1} \alpha_{t_i}\left(f(t_{i+1}) - f(t_i)\right)\Bigg]$$

Let $\varepsilon > 0$ and again by unif. cont.

Let a partition P s.t.

$$\left|\alpha_{t_i} - \alpha_{t_{i-1}}\right| \leq \frac{\varepsilon}{(f(b)-f(a))+1}$$

Supposing f is monotone increasing

$$M_i = f(x_i) \qquad m_i = f(x_{i-1})$$

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(\alpha_{t_i} - \alpha_{t_{i-1}})$$

$$\leq \frac{\varepsilon}{1+(f(b)-f(a))} \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$$

$$= \frac{\varepsilon}{1+(f(b)-f(a))}(f(b)-f(a)) < \varepsilon$$

Since ε was arbitrary $f \in \mathcal{R}(\alpha)$ $\square$

Thm Suppose f is bdd on $[a,b]$, f has at most finitely many pts of discontinuity on $[a,b]$ and α is cts at every pt where f is disc. then $f \in R(\alpha)$.

Before we prove this theorem lets see the "necessity" of the assumptions

For this we need to understand what discontinuous α means for the integral

... since α monotone can only have simple discontinuity

most basic example is

$$\alpha(x) = H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$ Heaviside function

Thm If $a < s < b$, f bdd on $[a,b]$ and f cts at s ~~then~~ and $\alpha(t) = H(t-s)$ then

$$\int_a^b f \, d\alpha = f(s)$$

I.e. $d\alpha$ is a Dirac δ mass at s.

proof: Take partitions $P = \{a, s-\delta, s+\delta, b\}$

~~$P = \{x_0, a, a+\delta, b\}$~~

Then $$U(P, f, \alpha) = \max_{s-\delta \le t \le s+\delta} f(t)$$

$$L(P, f, \alpha) = \min_{s-\delta \le t \le s+\delta} f(t)$$

Since f cts at s

$$U(P, f, \alpha),\ L(P, f, \alpha) \to f(s)$$

as $\delta \to 0$. $\square$

So integration against $d\alpha$ corresponds to evaluation at a point.

As you can see from the proof if f is not cts at s then $f \notin R(\alpha)$ on $[a,b]$

which makes, point evaluation doesn't really make sense for discontinuous functions.

Now lets return to the theorem on integrable functions

<u>proof</u> let $\varepsilon > 0$ and call $M = \max_{t \in [a,b]} |f(t)|$

let $E = \{\text{points of disc of } f\}$

E is finite and α cts on E

so we can choose finitely many <u>disjoint</u>

intervals $[u_j, v_j]$ s.t.

$$E \subset \bigcup_{j=1}^{N} (u_j, v_j) \quad , \quad \sum_{j=1}^{N} (\alpha_{v_j} - \alpha_{u_j}) < \varepsilon$$

Now on $K = [a,b] \setminus \bigcup_{j=1}^{N} (u_j, v_j)$ cpct

f is cts $\Rightarrow$ unif cts

$\Rightarrow$ $\exists\, \delta$ s.t. for $s, t \in K$ $|s-t| < \delta$

$\Rightarrow$ $|f(s) - f(t)| < \varepsilon$.

Now partition P of $[a,b]$ s.t.

all u_j, v_j in P

~~there~~ is $P \cap (u_j, v_j) = \phi \quad \forall\, j = 1, \dots, N$

and $x_i - x_{i-1} < \delta$ if x_i is not one of the ~~u_j~~ v_j

Then $M_i - m_i < \varepsilon$ if x_i is not a v_j

otherwise we use $M_i - m_i \le 2M$

$$U(P, f, \alpha) - L(P, f, \alpha) \le [(\alpha_b - \alpha_a)\varepsilon + 2M\varepsilon]$$

since ε arbitrary $f \in R(\alpha)$. $\square$

Thm Suppose $f \in R(\alpha)$ on $[a,b]$, $m \le f \le M$ and ϕ is cts on $[m, M]$ then $(\phi \circ f) \in R(\alpha)$ on $[a,b]$.

$\longrightarrow$

proof: let $\varepsilon > 0$, ϕ is unif. cts on $[m, M]$

so let $\delta > 0$ $\delta < \varepsilon$ WLOG s.t. $|u - v| < \delta \Rightarrow |\phi(u) - \phi(v)| < \varepsilon$

Then let P ~~$\delta > 0$~~ suff fine ~~small~~ s.t.

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$$

let M_i, m_i as usual and

$$M_i^* / m_i^* = \sup / \inf_{[t_{i-1}, t_i]} \phi \circ f$$

let two classes of subintervals in P

$i \in A$ if ~~$M_i^* - m_i^* < \delta$~~ $M_i - m_i < \delta$

$i \in B$ if $M_i - m_i \geq \delta$

for $i \in A$ $M_i^* - m_i^* < \varepsilon$ by choice of δ.

for $i \in B$ $M_i^* - m_i^* \leq 2 \sup_{[m, M]} |\phi|$ so

$$\delta \sum_{i \in B} (\alpha_{t_i} - \alpha_{t_{i-1}}) \leq \sum_{i \in B} (M_i - m_i) \cancel{(M_i^* - m_i^*)} (\alpha_{t_i} - \alpha_{t_{i-1}})$$

$$\leq \delta^2$$

Thus

$$U(P, \phi \circ f, \alpha) - L(P, \phi \circ f, \alpha) = \sum_{i \in A} (M_i^* - m_i^*)(\alpha_{t_i} - \alpha_{t_{i-1}}) + \sum_{i \in B} (M_i^* - m_i^*)(\alpha_{t_i} - \alpha_{t_{i-1}})$$

$$\leq \varepsilon(\alpha_b - \alpha_a) + 2 \sup_{[m, M]} |\phi| \, \delta < \varepsilon\left[(\alpha_b - \alpha_a) + 2 \sup |\phi|\right]$$