Properties of the R-S Integral

Thm (a) (Linearity) $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ then

$cf_1 \in \mathcal{R}(\alpha)$, $f_1 + f_2 \in \mathcal{R}(\alpha)$ for $c \in \mathbb{R}$ and

$$\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$$

$$\int_a^b cf_1\,d\alpha = c\int_a^b f_1\,d\alpha$$

(b) (Monotonicity) $f_1 \leq f_2$ on $[a,b]$ then

$$\int_a^b f_1\,d\alpha \leq \int_a^b f_2\,d\alpha$$

(c) (additivity) $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $a < c < b$

then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and $[c,b]$ and

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$$

(estimate (d)) $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $|f| \leq M$ on $[a,b]$

$$\left|\int_a^b f\,d\alpha\right| \leq M[\alpha(b) - \alpha(a)]$$

(e) $f \in R(\alpha_1)$ and $R(\alpha_2)$ then $f \in R(\alpha_1 + \alpha_2)$

$f \in R(\alpha)$, $c \in \mathbb{R}$ $\Rightarrow$ $f \in R(c\alpha)$

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proof:

(a) $f = f_1 + f_2$, P a partition of $[a,b]$

since $m_i(f) \geq m_i(f_1) + m_i(f_2)$

$M_i(f) \leq M_i(f_1) + M_i(f_2)$

$$L(f_1, P, \alpha) + L(f_2, P, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$$

suppose $f_1, f_2 \in R(\alpha)$ and let $\varepsilon > 0$

$\exists$ P_j $j = 1, 2$ s.t.

$$U(P_j, f_j, \alpha) - L(P_j, f_j, \alpha) < \varepsilon$$

$\Rightarrow$ same for $P = P_1 \cup P_2$

then $U(P, f, \alpha) - L(P, f, \alpha) \leq 2\varepsilon$

$\Rightarrow$ $f \in R(\alpha)$

then

$\mathrm{IU}\Sigma_1^n$

$$\left| \int_a^b f\,d\alpha - \int_a^b f_1\,d\alpha - \int_a^b f_2\,d\alpha \right| \le \left| \sum_{j=1}^{n} f(t_j)(\alpha_{t_j} - \alpha_{t_{j-1}}) - \sum_{j=1}^{n} f_1(t_j)(\alpha_{t_j} - \alpha_{t_{j-1}}) - \sum_{j=1}^{n} f_2(t_j)(\alpha_{t_j} - \alpha_{t_{j-1}}) \right|$$

$$+ \; 4\varepsilon$$

$$= 0 + 4\varepsilon$$

linearity of Riemann sums $\Rightarrow$ linearity of the integral.

other arguments are almost identical.

properties of sums $\rightsquigarrow$ properties of integral

for (c) just choose a partition P that contains the point c. $\square$

~~to get~~

Thm If $f, g \in R(\alpha)$ on $[a,b]$ then

(i) $f \cdot g \in R(\alpha)$ and

$$\int_a^b f g \, d\alpha \le \left(\int_a^b f^2 \, d\alpha \right)^{1/2} \left(\int_a^b g^2 \, d\alpha \right)^{1/2}$$

(Cauchy-Schwarz ineq)

(ii) $|f| \in R(\alpha)$ and

$$\left| \int_a^b f \, dt \right| \le \int_a^b |f| \, dt$$

(Triangle ineq)

<u>proof</u>: for (i) ~~take~~ note

$$\underbrace{(f+g)^2 - (f-g)^2}_{\in R(\alpha)} = 4fg$$

since we know

cts fcn of Riemann integrable $\Rightarrow$ Riemann integrable.

for (ii), take $\phi(t) = |t|$ to see $|f| \in R(\alpha)$

$$\left| \int_a^b f \, d\alpha \right| = \operatorname{sgn}\left(\int_a^b f \, d\alpha \right) \int_a^b f \, d\alpha = \int_a^b \operatorname{sgn}\left(\int_a^b f \, d\alpha \right) f \, d\alpha \le \int_a^b |f| \, d\alpha .$$

Thm Assume α monotone and $\alpha' \in \mathcal{R}$ on $[a,b]$.

Let f be a bounded real valued fcn on $[a,b]$

Then $f \in \mathcal{R}(\alpha) \iff f\alpha' \in \mathcal{R}$ and

$$\int_a^b f \, d\alpha = \int_a^b f \alpha' \, dt$$

Proof: Let $\varepsilon > 0$, $\alpha' \in \mathcal{R}$ so $\exists$ partition P w/

$$U(P, \alpha') - L(P, \alpha') < \varepsilon$$

By MVT $\alpha_{t_i} - \alpha_{t_{i-1}} = \alpha'(x_i)(t_i - t_{i-1})$ for some $x_i \in (t_{i-1}, t_i)$

Then for any choices $s_i \in (t_{i-1}, t_i)$

(*) $\left| \sum_{i=1}^{n} f(s_i)(\alpha_{t_i} - \alpha_{t_{i-1}}) - \sum_{i=1}^{n} f(s_i)\alpha'(s_i) \cancel{(\alpha_{t_i} - \alpha_{t_{i-1}})} (t_i - t_{i-1}) \right|$

$$\leq \max_{[a,b]} |f| \sum_{i=1}^{n} |\alpha'(x_i) - \alpha'(s_i)| (t_i - t_{i-1})$$

$$\leq \max_{[a,b]} f \left(U(P, \alpha') - L(P, \alpha') \right) \leq \underbrace{\left(\max_{[a,b]} |f| \right)}_{=M} \varepsilon$$

So from (*)

$$U(P, f, \alpha) \leq \cancel{U(f\alpha} \; U(P, f\alpha') + M\varepsilon$$

and $$U(P, f\alpha') \leq U(P, f, \alpha) + M\varepsilon \rightarrow$$

~~Same for lower sums~~

$$\Rightarrow \quad \overline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\alpha'\, dt$$

(since ε arbitrary)

and same for lower integrals

for <u>any</u> bounded f.

Now if $f\alpha' \in R$ then (and only then)

$$\overline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\alpha'\, dt = \underline{\int_a^b} f\alpha'\, dt = \underline{\int_a^b} f\, d\alpha \quad \square$$

<u>Thm</u> (Change of Variables)

Suppose φ is cts strictly increasing ~~$\varphi: [a,b] \to$~~ φ maps $[A,B]$ onto $[a,b]$. Suppose $f \in R(\alpha)$ on $[a,b]$. Define β, g on $[A,B]$ by

$$\beta(y) = \alpha(\varphi(y)), \quad g(y) = f(\varphi(y))$$

Then $g \in R(\beta)$ and

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha .$$

example: $[a,b] = [1,2]$, ~~$\varphi(x) =$~~

$\int_1^2 f(x)\,dx$ want to change variables to

$u = x^2$

(e.g. maybe have $\boxed{\int_1^2 2x e^{-x^2}\,dx}$)

So $[A,B] = [1,4]$

$\varphi(u) = u^{1/2}$, $d\varphi = \frac{1}{2}\frac{1}{u^{1/2}}\,du$

$$\int_a^b f(x)\,dx = \int_A^B f(\varphi(u))\,d\varphi(u) = \int_1^4 f(u^{1/2})\frac{1}{2u^{1/2}}\,du$$

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proof: given a partition P of $[a,b]$ let Q be the partition of $[A,B]$ given by

$\varphi(u_i) = t_i \in P$.

Any partition Q of $[A,B]$ can be obtained in this way

since $f([t_{i-1}, t_i]) = g([u_{i-1}, u_i])$

$U(P, f, \alpha) = U(Q, g, \beta)$, $L(P, f, \alpha) = L(Q, g, \beta)$

$\beta_{u_i} - \beta_{u_{i-1}} = \alpha_{\varphi(u_i)} - \alpha_{\varphi(u_{i-1})} = \alpha_{t_i} - \alpha_{t_{i-1}}$

since $f \in R(\alpha)$ can choose P to make

$U(P, f, \alpha)$, $L(P, f, \alpha)$ close to

$\int_a^b f\,d\alpha \Rightarrow$

$g \in R(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$ $\square$

e.g. for $\varphi \in C'$ (or $\varphi' \in R$ on $[A, B]$)

$$\boxed{\int_a^b f\,dx = \int_A^B f(\varphi(y))\,\varphi'(y)\,dy}$$

Change of variables formula

Fundamental Thm of Calculus

Thm Let $f \in \mathcal{R}$, bounded, on $[a,b]$ for $a \le x \le b$ define

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is Lipschitz cts on $[a,b]$, if f is cts at $x_0 \in [a,b]$ then F is diff at x_0 and

$$\boxed{F'(x_0) = f(x_0)}$$

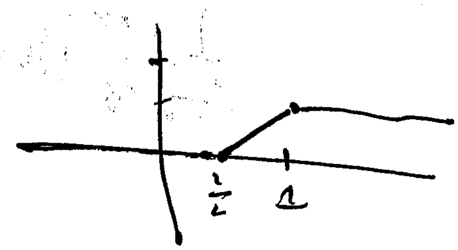
example

$$f(t) = \mathbb{1}_{[1/2,1]}(t) = \begin{cases} 1 & 1/2 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

indicator function

f has jump discontinuities at $\frac{1}{2}$, 1 cannot be the derivative of any everywhere differentiable fn F in a nbhd of those pts (IVT for derivatives of everywhere diff fns)

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} 0 & t \le \frac{1}{2} \\ t - \frac{1}{2} & \frac{1}{2} < t \le 1 \\ \frac{1}{2} & t \ge 1 \end{cases}$$



Proof: $f \in R \Rightarrow$ bdd let M s.t. $|f| \le M$ on $[a,b]$

for $a \le y \le x \le b$

$$|F(y) - F(x)| = \left|\int_y^x f(t)\,dt\right| \le M|x-y|$$

$\Rightarrow F$ is Lipschitz

~~Fix~~ for any ~~x~~ $s \le x_0 \le t$

$$\left|\frac{F(t)-F(s)}{t-s} - f(x_0)\right| = \left|\frac{1}{t-s}\int_s^t f(u)\,du - \frac{1}{t-s}\int_s^t f(x_0)\,du\right|$$

$$= \left|\frac{1}{t-s}\int_s^t [f(u) - f(x_0)]\,du\right|$$

$$\le \frac{1}{t-s}\int_s^t |f(u) - f(x_0)|\,du$$

let $\varepsilon > 0$ and $\delta > 0$ s.t.

$$|u - x_0| < \delta \Rightarrow |f(u) - f(x_0)| < \varepsilon$$

then for $x_0 - \delta < s \le x_0 \le t \le x_0 + \delta$

$$\frac{1}{t-s}\int_s^t |f(u) - f(x_0)|\,du \le \frac{\varepsilon}{t-s}\int_s^t 1\,du = \varepsilon \quad \square$$

Thm (Fund. Thm of Calc.) If $f \in \mathcal{R}$ on $[a,b]$ and if $f = F'$ for some diff F on $[a,b]$ then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

e.g. applies to C^1 F.

~~proof:~~ (Restated)

Thm If F is diff on $[a,b]$ and $F' \in \mathcal{R}$ on $[a,b]$

then $$\int_a^b F'(t)\,dt = F(b) - F(a).$$

proof: let $\varepsilon > 0$, let partition P s.t.

$$U(P, F') - L(P, F') < \varepsilon.$$

Then by the MVT

$$F(t_i) - F(t_{i-1}) = F'(x_i)(t_i - t_{i-1}) \quad \text{for some } t_{i-1} \le x_i \le t_i$$

$$\sum_{i=1}^n F'(x_i)(t_i - t_{i-1}) = \sum_{i=1}^n F(t_i) - F(t_{i-1}) = F(b) - F(a)$$

so $$\left| \int_a^b F'(t)\,dt - \big(F(b) - F(a)\big) \right| < \varepsilon \quad \square$$

Thm (Integration by Parts)

Suppose F, G diff on $[a,b]$, $F' = f \in R$

$G' = g \in R$ on $[a,b]$. Then

$$\int_a^b F g \, dt = (FG)\Big|_a^b - \int_a^b G f \, dt$$

proof: Apply fundamental thm of Calc to $H(t) = F(t)G(t)$,

use product rule and that

$$H' = Gf + Fg \quad \in R \quad \text{on } [a,b]$$

(product of Riemann integrable fcns) ∎

Vector Valued Fcns

suppose $\vec{f} : [a,b] \to \mathbb{R}^k$

$$\vec{f}(t) = (f_1(t), \ldots, f_k(t))$$

We say $f \in R(\alpha)$ on $[a,b]$ if all $f_k \in R(\alpha)$ on $[a,b]$.

$$\boxed{\int_a^b \vec{f} \, d\alpha = \left(\int_a^b f_1 \, d\alpha, \ldots, \int_a^b f_k \, d\alpha\right)}$$

Thm: If $\vec{f}:[a,b]\to\mathbb{R}^k$ diff on $[a,b]$ and $f' \in R$

then $\int_a^b \vec{f}'(t)\,dt = \vec{f}(b)-\vec{f}(a)$

proof: Apply FTC to component ~~fctns~~ functions

Thm If $f:[a,b]\to\mathbb{R}^k$, $f\in R(\alpha)$ for some monoton

α, then $|f|\in R(\alpha)$ and

$$\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$$

proof: Note $|f| = (f_1^2+\dots+f_k^2)^{1/2}$

is a cts fcn of $R(\alpha)$ fns $\Rightarrow$ it is $R(\alpha)$

Call $w_j = \int_a^b f_j\,d\alpha \Big/ \left|\int_a^b f\,d\alpha\right|$, $\vec{w}=(w_1,\dots,w_k)$, $|\vec{w}|=1$

~~$|w|^2 = \sum_1^k w_j^2 = \sum_1^k w_j \int_a^b$~~

(if $\left|\int_a^b f\,d\alpha\right| = 0$ result is trivial)

$$\left|\int_a^b f\,d\alpha\right| = w\cdot\int_a^b f\,d\alpha \le \int_a^b |w||f|\,d\alpha = \int_a^b |f|\,d\alpha.$$

Bounded variation functions

If we take $\alpha = \beta - \gamma$ w/

β, γ both monotone increasing

we can define $f \in R(\alpha)$ by $f \in R(\beta) \cap R(\gamma)$

$$\int_a^b f \, d\alpha := \int_a^b f \, d\beta - \int_a^b f \, d\gamma$$

If α is the difference of two monotone functions we say f is <u>bounded variation</u>

why this name?

let $t \in [a,b]$ given any partition P of $[a,t]$ define

$$\Lambda(P, \alpha) = \sum_{i=1}^{n} |\alpha_{t_i} - \alpha_{t_{i-1}}|$$ (curve length) <u>total variation</u>

define $L(t) = \sup_P \Lambda(P, \alpha)$ which is clearly monotone increasing. If $L(t) < +\infty$ on $[a,b]$ we say α has <u>finite total variation</u>

consider $\beta(t) = L(t) - \alpha(t)$

Claim: β is bounded and monotone increasing on $[a,b]$

let $a \le t_1 \le t_2 \le b$

$$\beta(t_2) - \beta(t_1) = \cancel{L(t_2) - L(t_1)}$$

$$L(t_2) - \alpha(t_2) - L(t_1) + \alpha(t_1)$$

$$\cancel{= L(t_2) - L(t_1) + \alpha(t_2) - \alpha(t_1)}$$

$$= L(t_2) - [\,L(t_1) + \alpha(t_2) - \alpha(t_1)\,]$$

$$\ge L(t_2) - [\,L(t_1) + |\alpha(t_2) - \alpha(t_1)|\,]$$

$$\ge L(t_2) - L(t_2) = 0$$

so β is monotone increasing.

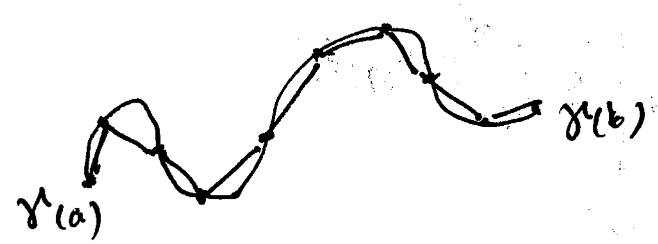
Length of Curves (Rectifiability)

Def A continuous mapping γ of an interval $I \subset \mathbb{R}$ into $\mathbb{R}^k$ is called a <u>curve</u> or <u>path</u>

If γ is 1-1 we say γ is an <u>arc</u>

If $\gamma(a) = \gamma(b)$ we say γ is a <u>closed curve</u>

How to compute the length of a curve?



given a partition P of $[a,b]$ define

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})|$$

Then the <u>length</u> of γ is defined

$$\boxed{\text{length}(\gamma) = \sup_P \Lambda(P, \gamma)}$$

formally $\quad \text{length}(\gamma) = \int_0^b |d\gamma|$

if $\quad \text{length}(\gamma) < +\infty \quad$ we say γ rectifiable

or finite length

Thm If γ' is cts on $[a,b]$ then γ

rectifiable and

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)|\,dt .$$

proof: for any $a \le t_{i-1} \le t_i \le b$

$$|\gamma(t_i) - \gamma(t_{i-1})| = \left|\int_{t_{i-1}}^{t_i} \gamma'(s)\,ds\right| \le \int_{t_{i-1}}^{t_i} |\gamma'(s)|\,ds$$

so for any partition P

$$\Lambda(P,\gamma) \le \int_a^b |\gamma'(t)|\,dt < +\infty$$

so γ is rectifiable

Now let's go the other direction. let $\varepsilon > 0$

$\exists \ \delta > 0$ st. $|\gamma'(t) - \gamma'(s)| < \varepsilon$ when $|t-s| < \delta$

let P be a partition of $[a,b]$ w/

$t_i - t_{i-1} \le \delta \quad \forall \quad 1 \le i \le n$.

so $$\int_{t_{i-1}}^{t_i} |\gamma'(t)|\,dt \leq |\gamma'(t_i)|(t_i - t_{i-1}) + \varepsilon(t_i - t_{i-1})$$

$$= \left|\int_{t_{i-1}}^{t_i} \gamma'(t_i)\,dt\right| + \varepsilon(t_i - t_{i-1})$$

$$\leq \left|\int_{t_{i-1}}^{t_i} \gamma'(t)\,dt\right| + \left|\int_{t_{i-1}}^{t_i} \gamma'(t_i) - \gamma'(t)\,dt\right| + \varepsilon(t_i - t_{i-1})$$

$$\leq |\gamma(t_i) - \gamma(t_{i-1})| + 2\varepsilon(t_i - t_{i-1})$$

summing up we get

$$\int_a^b |\gamma'(t)|\,dt \leq \Lambda(P, \gamma) + 2\varepsilon(b-a)$$

$$\leq \Lambda(\gamma) + 2\varepsilon(b-a)$$

Since ε was arbitrary

$$\int_a^b |\gamma'(t)|\,dt \leq \Lambda(\gamma).$$

Thus we have both inequalities

$$\int_a^b |\gamma'(t)|\,dt = \Lambda(\gamma). \quad \square$$