MATH 204: Homework 4
Due Wed Feb 1

Problems are from Rudin 3rd edition.
Problem 1. Chapter 9 (p. 239): 21

Problem 2. Let $E \subset \mathbb{R}^n$ open and $f : E \to \mathbb{R}$ be C^1. In class we saw that $f'(x) \in L(\mathbb{R}^n, \mathbb{R})$ can be represented as $f'(x)h = \nabla f(x) \cdot h$ for a vector $\nabla f(x) \in \mathbb{R}^n$ which we called the gradient of f. This makes ∇f a mapping from E to $\mathbb{R}^n$. Suppose that ∇f is itself differentiable on E and call it's derivative $D^2 f : E \to L(\mathbb{R}^n)$.

(a) Suppose that f has a local maximum at a point $x \in E$. Show that $D^2 f(x) \leq 0$ in the sense that,

$$\xi \cdot D^2 f(x)\xi \leq 0 \quad \text{for every} \quad \xi \in \mathbb{R}^n.$$

A linear operator with such a property is called *non-positive definite* (or *negative definite* if there is strict inequality).

(b) You saw on the midterm that if f has an interior local maximum at a point $x \in E$ then $\nabla f(x) = 0$. Now let us suppose that the domain $E = \{x \in \mathbb{R}^n : g(x) < 0\}$ for a C^1 function g which satisfies $\nabla g \neq 0$ on ∂E. Here we will need that both f and g are actually C^1 on an open set containing the closure of E. Suppose that f attains its maximum over the set $\overline{E}$ at a point $x \in \partial E$. Show that,

$$\nabla f(x) \cdot \nu(x) \geq 0,$$

where $\nu(x) = \frac{\nabla g(x)}{|\nabla g(x)|}$ is the unit normal vector to the domain E at the point x.

(c) In the same setting as part (b) assume that ∇f is also differentiable at the maximum point $x \in \partial E$. Let T_x be the orthogonal complement of the vector $\nu(x)$ in $\mathbb{R}^n$, i.e. T_x is the subspace of directions tangential to ∂E at x. Show that,

$$\xi \cdot D^2 f(x)\xi \leq 0 \quad \text{for every} \quad \xi \in T_x,$$

i.e. $D^2 f$ is non-negative definite in the tangential directions to ∂E.

(d) Now lets consider the values of f restricted to $\partial E = \{x \in \mathbb{R}^n : g(x) = 0\}$. Suppose that f attains $\max_{\partial E} f$ at a point $x \in \partial E$. Finding the value/location of such a maximum is referred to as a *constrained optimization problem*. Show that at such a point,

$$\nabla f(x) = \lambda \nabla g(x) \quad \text{for some} \quad \lambda \in \mathbb{R}.$$

The parameter λ is often referred to as a *Lagrange Multiplier*.