Higher Derivatives

If f' exists on an interval around a point x we can define $f''(x)$ etc.

nth derivative $f^{(n)}(x)$.

f is <u>n times differentiable</u> at x if it has $(n-1)$ derivatives existing in a neighborhood of x and $f^{(n-1)}$ is diff. at x.

<u>Taylor's Thm</u> Suppose $f: [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}(t)$ exists for all $t \in (a,b)$. Then for every $x, x_0 \in [a,b]$

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$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n-1)}}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(t)}{n!}(x-x_0)^n$$

for some t between x, x_0

proof:

Call $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!} (t - x_0)^k$

$(n-1)$ order Taylor polynomial

Call M s.t.

$f(x) = P(x) + M(x - x_0)^n$ ~~and $g(t) = f(t) - P(t) - M(t-x_0$~~

~~note that $f^{(k)}$~~

and $g(t) = f(t) - P(t) - M(t-x_0)^n$

$g^{(n)}(t) = f^{(n)}(t) - n!M$

and $g^{(k)}(x_0) = f^{(k)}(x_0) - f^{(k)}(x_0) = 0$

so now $g(0) = 0$ and M chosen s.t. $g(x) = 0$

$\Rightarrow \exists \; x_1 \in (x_0, x)$ s.t. $g'(x_1) = 0$

then $g'(x_0) = g'(x_1) = 0 \Rightarrow \exists \; x_2 \in (x_0, x_1)$

w/ $g''(x_2) = 0$

etc ...

$\exists \quad x_k \in (x_0, x_{k-1}) \quad$ s.t. $\quad g^{(k)}(x_k) = 0$

for $\quad k = ~~\text{expn}~~ 1, 2, \ldots, n$

but $\quad g^{(n)}(x_n) = 0$

$$\Rightarrow \quad f^{(n)}(x_n) = n!\,M$$

$$\Rightarrow \quad M = \frac{f^{(n)}(x_n)}{n!} \quad , \quad \text{QED}$$

Differentiation of vector Valued fns on $\mathbb{R}$

We can apply the same definition of differentiable for vector valued fns

for example $\quad f : [a,b] \to \mathbb{C}$

or $\quad f : [a,b] \to \mathbb{R}^k$

$f'(x) \in \mathbb{R}^k$ is defined s.t.

$$\lim_{t \to x} \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = 0$$

note that limit holds iff. each component

converges

$$\frac{f_j(t)-f_j(x)}{t-x} \longrightarrow \cancel{\frac{f_j'(t)-f_j'(x)}{t-x}} \quad f_j'(x) \quad \text{as } t \to x$$

i.e. $f : [a,b] \to \mathbb{R}$ is diff. at $x \in [a,b]$

iff $\forall$ $1 \leq j \leq k$ the component fcns

$f_j(t)$ are diff. at x, and

$$f'(x) = (f_1'(x), \ldots, f_k'(x)).$$

<u>Thm</u> f diff $\Rightarrow$ f cts

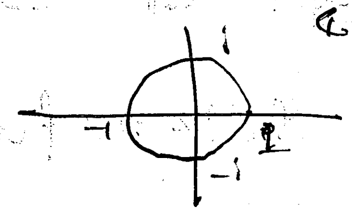
<u>Thm</u> $(f+g)' = f' + g'$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

The mean value theorem however does not hold for vector valued fcns.

Example: $f(x) = e^{ix} = \cos x + i \sin x$

on $[0, 2\pi]$



this is a paramatrization of the unit circle

$|f(x)|^2 = \cos^2 x + \sin^2 x = 1$ for all $x \in \mathbb{R}$.

$f(0) = f(2\pi) = 0$

but $f'(x) = \sin x - i \cos x = i e^{ix}$

also has $|f'(x)| = 1$ for all x

so $f'(x) \neq 0$ for any $x \in [0, 2\pi]$

On the other hand for $f: [a,b] \to \mathbb{R}^k$ (or $\mathbb{C}$) $|f(x)|^2$ does satisfy the MVT since it is a map $[a,b] \to \mathbb{R}$ which can sometimes be useful

Even more:

Thm Suppose f is a cts map $[a,b]$ to $\mathbb{R}^k$ and f is diff on (a,b). Then $\exists x \in (a,b)$ s.th

$$|f(b) - f(a)| \leq (b-a)|f'(x)|.$$

proof: Call $z = f(b) - f(a) \in \mathbb{R}^2$ and define

$\varphi(t) = z \cdot f(t)$ for $a \leq t \leq b$

this is cts on $[a,b]$ and diff on (a,b) as a finite sum of diff. functions

Then MVT $\Rightarrow$

$$\varphi(b) - \varphi(a) = (b-a)\varphi'(x) = (b-a)\, z \cdot f'(x) \quad \text{for some } x \in (a,b)$$

$$\varphi(b) - \varphi(a) = z \cdot (f(b) - f(a)) = |f(b) - f(a)|^2$$

so $|f(b) - f(a)|^2 = (b-a)\, z \cdot f'(x)$

$\leq (b-a)\,|z|\,|f'(x)|$

so $|f(b) - f(a)| \leq (b-a)\,|f'(x)|$ for some $x \in (a,b)$ $\square$

Ordinary Differential Equations

ODE initial value problem look for $y : [t_0, T] \to \mathbb{R}^k$

solving $\left[\begin{array}{l} y' = \phi(t, y) \quad \text{for } t > t_0 \\ y(t_0) = y_0 \end{array}\right.$ $\qquad f : [t_0, T] \times \mathbb{R}^k \to \mathbb{R}^k$

~~2~~ questions: 1. Existence of solutions
2. Uniqueness of solutions

~~Thm~~ Suppose that $\phi : [t_0, T] \times \mathbb{R}^k \to \mathbb{R}^k$ is Lipschitz cts in y variable.
~~Thm~~ There exists at most one $y : [t_0, T]$

which is cts on $[t_0, T]$ and diff on (t_0, T)
solving $\left[\begin{array}{l} y'(t) = \phi(t, y(t)) \quad \text{for } \cancel{t > t_0} \ t \in (t_0, T) \\ y(t_0) = y_0 \end{array}\right.$

proof: suppose y_1 and y_2 both solve

then $\left[\begin{array}{l} (y_1 - y_2)' = \cancel{\phi(t, y_1)} \ \phi(t, y_1) - \phi(t, y_2) \\ (y_1 - y_2)(t_0) = 0 \end{array}\right.$

~~by the "mean value property" for vector valued fns~~

~~then~~ $\to$ $\cancel{y_1}$

Call $z = y_1 - y_2$

$$|z'| = |\phi(t, y_1) - \phi(t, y_2)| \leq A|z|$$

using that ϕ Lipschitz in y variable.

thus z solves

$$\begin{cases} |z'| \leq A|z| & \text{for } t \in [t_0, T] \\ z(t_0) = 0 \end{cases}$$

Lem: Suppose z solves the above eqn. Then $z(t) = 0$ for all $t \in [t_0, T]$

proof: Call $M = \max_{[t_0, T]} |z|$

and let $\delta > 0$ small enough that

$$\delta A < 1 .$$

We show that $z(t) = 0$ for $t \in [t_0, t_0 + \delta]$ then just apply same argument $\left\lceil \frac{T - t_0}{\delta} \right\rceil$ times

by vector MVP for every $t \in [t_0, t_0 + \delta]$

$\exists\ \tau \in (t_0, t)$ such that

$$\begin{aligned} |z(t)| &\leq (t - t_0)|z'(\tau)| \\ &\leq \delta A |z(\tau)| \\ &\leq \delta A M \end{aligned}$$

since t was arbitrary

$$M \le \underset{\text{if } M \ne 0}{SAM < M} \quad \text{a contradiction if } M \ne 0$$

$$\Rightarrow \max_{t \in [t_0, t_0+\delta]} |z(t)| = 0 \qquad \square$$

We will return to the question of existence later,

we need integration

(e.g. we can even solve the "trivial" ODE

$$\begin{cases} y' = f(t) \\ y(t_0) = y_0 \end{cases}$$ yet.