~~Analysis in $\mathbb{R}^n$, II 2~~

Analysis in $\mathbb{R}^n$ (Math 204)

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Homework assigned (on my webpage) weekly on Wednesdays.

2 Hour Exams	+ Final	+ HW
20% each	50%	10%

Homework is essential part of learning ~~despite/because~~

this is reason for low grade %.

(i.e. don't copy solutions for HW!)

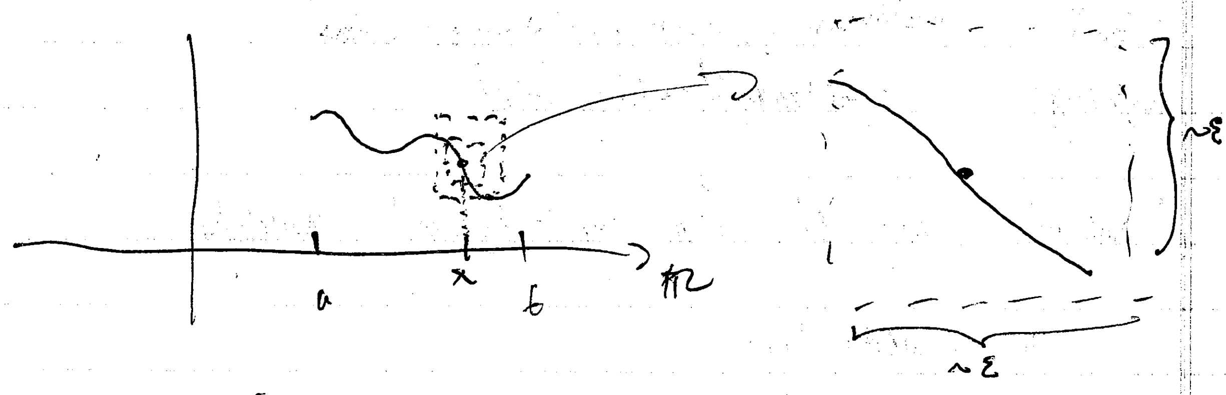
Course Plan: Rudin Ch 5, 9, 6, 7

- Differentiation in 1-variable
- Differentiation on $\mathbb{R}^n$
  - Inverse and Implicit function Thms.

- Riemann - Stieltjes Integration
- Sequences of functions ~~convergence~~

Differentiation on $\mathbb{R}$

let a function $f : [a,b] \to \mathbb{R}$ which is "regular"



zoom in at a point x, "regular" enough function looks more and more linear

how to make this precise? want to find a

linear function $l : \mathbb{R} \to \mathbb{R}$ so that

$$f(y) = l(y) + g(y)$$

with $$\lim_{y \to x} \frac{g(y)}{|y - x|} = 0$$

(i.e. $g(x) = 0$ and g goes to zero faster than any linear function at x)

from this we see $\ell(x) = f(x)$

so must be $\boxed{\ell(y) = f(x) + a(y-x)}$

for some $a \in \mathbb{R}$.

how to find this a?

supposedly $f(y) - f(x) = a(y-x) + g(y)$

and $\lim_{y \to x} \frac{f(y) - f(x)}{y-x} = a + 0$

$= a$

so the slope of this linear function

should be the "slope" of the graph of f at x.

Starting from this limit define

the <u>derivative</u> of f at x to be

$f'(x) := \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ if said limit exists

(in the sense we defined last quarter)

If f' exists at every point of a set $E \subset [a,b]$ we say f is <u>differentiable</u> on E.

Can also work on open interval (a,b) but then $f'(a)$ and $f'(b)$ not defined.

<u>Thm</u> If f is diff. ~~then~~ at x then f is cts at x.

<u>proof</u>:

$$f(t) - f(x) = (t-x)\frac{f(t)-f(x)}{t-x} \to f'(x)\cdot 0 = 0.$$

as $t \to x$

Using that limit of product ~~and~~ is product of limits. $\square$

On the other hand cts $\not\Rightarrow$ differentiable

(e.g. $\sqrt{|x|}$ at $x=0$)

We can "construct" later continuous functions which are nowhere differentiable!

Thm Spse f, g def on $[a,b]$ and diff'able at $x \in [a,b]$. Then

(a) $(f+g)'(x) = f'(x) + g'(x)$

(b) $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

(iii) quotient rule. (Corollary of composition rule)

proof: (a) is just property of limits

(b) call $h = f \cdot g$

$$\frac{h(t) - h(x)}{t - x} = \frac{f(t)g(t) - f(t)g(x) + f(t)g(x) - f(x)g(x)}{t - x}$$

$$= f(t)\frac{g(t) - g(x)}{t-x} + g(x)\frac{f(t) - f(x)}{t-x}$$

$$\to f(x)g'(x) + g(x)f'(x)$$

since f, g diff'able at x and (hence) Cts at x also. $\square$

Examples

- $f(x) = a$ constant

$f'(x) = 0$

- $f(x) = \cancel{ax}\ a+bx$ linear

$$f'(x) = 0 + \lim_{t\to x} \frac{a(t-x)}{(t-x)} = 0 + a = a$$

- $f(x) = x^n$ for any integer $n \neq 0$

$n>0$

$$(x+h)^n - x^n = \cancel{x^n} + nx^{n-1}h + \cdots + h^n \cancel{-x^n}$$

$$\text{so} \quad \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}$$

$f'(x) = nx^{n-1}$ (for $n<0$ use quotient rule)

~~Chain~~

Thm (Chain Rule) Suppose f is cts on $[a,b]$ and diff. at $x \in [a,b]$ and g is defined on an interval I containing ~~range~~ $f([a,b])$, and g is diff at $f(x)$. If $h(t) = g(f(t))$ then h is diff at x and $h'(x) = f'(x)g'(f(x))$.

Examples:

- $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ continuous at $x=0$ but not diff'able

$$\frac{f(t) - f(0)}{t - 0} = \frac{t \sin \frac{1}{t} - 0}{t - 0} = \sin \frac{1}{t}$$

does not converge as $t \to 0$.

- $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ cts at 0 and

$$f'(0) = \lim_{t \to 0} \frac{t^2 \sin \frac{1}{t} - 0}{t - 0} = \lim_{t \to 0} t \sin \frac{1}{t} = 0$$

limit exists.

Mean Value Thm

Def: We say $f: X \to \mathbb{R}$ (metric space) has a local max at $x \in X$

if $\exists$ $\delta > 0$ s.t. for all $d(x,y) < \delta$

$f(y) \leq f(x)$.

proof: Let $y = f(x)$, from the definition of the derivative

$$f(t) - f(x) = (t-x)[f'(x) + u(t)] \qquad t \in [a,b]$$

$$g(s) - g(y) = (s-y)[g'(y) + v(s)] \qquad s \in I$$

w/ $\quad u(t) \to 0 \quad$ as $\quad t \to x$

$\quad v(s) \to 0 \quad$ as $\quad s \to y$

~~Let~~ Then

$$h(t) - h(x) = g(f(t)) - g(f(x))$$

$$= (f(t) - f(x))[g'(f(x)) + v(f(t))]$$

$$= (t-x)[f'(x) + u(t)][g'(f(x)) + v(f(t))]$$

now $\quad u(t) \to 0 \quad$ as $\quad t \to x$

$\quad f(t) \to f(x) \quad$ as $\quad t \to x$

$\Rightarrow \quad v(f(t)) \to v(f(x)) = 0 \quad$ as $\quad t \to x$

so $$\lim_{t \to x} \frac{h(t) - h(x)}{t - x} = f'(x)\, g'(f(x)) \qquad \square$$

Thm If $f : [a,b] \to \mathbb{R}$ has a local max at $x \in (a,b)$ and $f'(x)$ exists then

$$f'(x) = 0.$$

Rem: if $x = b$ then $f'(x) \geq 0$

$x = a$ then $f'(x) \leq 0$

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proof: let $\delta > 0$ s.t. $a < x - \delta < x < x + \delta < b$

and $f(x) \geq f(t) \quad \forall\, t \in [x-\delta, x+\delta]$

Then

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$$= \lim_{t \searrow x} \frac{f(t) - f(x)}{t - x} \leq 0$$

Similarly taking limit as $t \nearrow x$

$$f'(x) \geq 0$$

$$\Rightarrow \quad f'(x) = 0.$$

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<u>Thm</u> If f, g are cts real valued fcn on $[a,b]$ diff. on (a,b) Then $\exists\, x \in (a,b)$ such that

$$(f(b)-f(a))g'(x) = (g(b)-g(a))f'(x)$$

<u>Cor</u> f as above then $\exists\, x \in (a,b)$ s.t.

$$f(b)-f(a) = (b-a)f'(x) .$$

Pf: let $h(t) = (f(b)-f(a))g(t) - (g(b)-g(a))f(t)$

h is cts on $[a,b]$, diff on (a,b)

and $h(a) = h(b) = f(b)g(a) - g(b)f(a)$

thus h is either (1) constant or

(2) has a local max or min on

in case (1) take any $x \in (a,b)$

$h'(x) = 0$ is the result.

if x not constant $\exists\, x \in (a,b)$ s.t. $h(a)$

(take $h(x) > h(a)$ (or $h($) WLOG

and so (since cts functions achieve extreme on cpct set)

$\exists$ x in $[a,b]$ s.t.

$$h(x) = \max_{t\in[a,b]} h(t) > h(a) = h(b)$$

so $x \in (a,b)$

by previous thm $h'(x) = 0$
which is the desired result.

<u>Thm</u>: Suppose $f : (a,b) \to \mathbb{R}$ differentiable

(a) If $f'(x) \geq 0$ for all $x \in (a,b)$ then f is monotone increasing

(b) If $f'(x) = 0$ for all $x \in (a,b)$ then f is monotone decreasing

<u>proof</u>: for each pair $x_1, x_2 \in (a,b)$

$$f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$$

for some $x \in (x_1, x_2)$

~~just because~~ $f'(x)$ may exists at every point but be a discontinuous function

e.g. $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

We showed last time

$f'(0) = 0$, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ for $x \neq 0$

however a function which is differentiable at every point of an interval still has some continuity property from the MVT. In particular it satisfies the consequence of IVP for cts functions

Thm Suppose f diff on $[a,b]$ and $f'(a) < \lambda < f'(b)$ then there is $x \in (a,b)$ such $f'(x) = \lambda$.

proof: put $g(t) = f(t) - \lambda t$ then

$g'(a) = f'(a) - \lambda < 0$ and $g'(b) = f'(b) - \lambda > 0$

so that $\min_{t \in [a,b]} g(t)$ cannot be attained

on $\{a, b\}$. Let $t_0 \in (a,b)$

where minimum is attained. Then

$$0 = g'(t_0) = f'(t_0) - \lambda$$ which is

what we wanted ∎

In particular if a function f is diff on an interval it cannot have a simple discontinuity there.

(although as we've seen more general discontinuities are possible)

L'hopital's Rule

you probably remember this "rule" about calculating limits $\lim_{x \to a} \frac{f(x)}{g(x)}$ when f and g both converge to 0 or $\pm\infty$. ~~or both~~

it's a bit misleading to say that one should apply a rule in these situations, it is really more of an art since you will run into similar situations a lot with slightly different set up. The idea is to see if the growth rate or rate of convergence to zero match in the numerator and denominator examples :

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x(1+u(x))}{x} = \lim_{x\to 0}\left[1+u(x)\right] = 1$$

Using that (from defn of differentiable) and $(\sin x)'(0) = 1$

$$\sin x = x(1+u(x)) \quad \text{w/ } u(x)\to 0 \text{ as } x\to 0$$

Thm: Suppose f, g diff. on (a,b) and $g'(x)\neq 0$ on (a,b), w/ $-\infty \le a < b \le +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \quad \text{as } x\to a$$

If $f(x), g(x) \to 0$ as $x\to a$

or $g(x)\to +\infty$ as $x\to a$

then $\frac{f(x)}{g(x)} \to A$ as $x\to a$.

proof: again don't take these particular assumptions too seriously, this is just an example of a general idea.

I'll just prove the simpler case $f(x), g(x) \to 0$ as $x \to a$

Consider the case $-\infty \leq A < +\infty$ and let any $q > A$. We want to show there is $c \in (a, b)$ such that $a < y < c$ implies

$$\frac{f(y)}{g(y)} \cancel{\text{?}} < q$$

Let $A < r < q$ there is c ~~s.t.~~ $\in (a, b)$ s.t.

$$a < x < c \Rightarrow \frac{f'(x)}{g'(x)} < r$$

If $a < x < y < c$ then MVT implies $\exists$ $t \in (x, y)$ s.t.

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r$$

then let $x \to a$ we get

$$\frac{f(y)}{g(y)} \leq r < q \quad \text{for all } a<y<c,$$

for the other direction if $-\infty < A \leq +\infty$

we argue as before for any $p<A$,

show that there is $d \in (a,b)$ s.t.

$$a<x<d \Rightarrow \frac{f(x)}{g(x)} > p \quad \Box$$