The Inverse function Theorem

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~~Recall that a linear~~

~~The inverse function theorem says that if a~~

~~function~~ $\cancel{f : E \to \mathbb{R}^n}$ ~~is diff~~ $\subset \mathbb{R}^n$

We now aim to prove two important theorems about differentiable functions

1. Inverse function thm
2. Implicit function thm

~~The first~~ 1. says that if

$f'(x)$ is invertible then f is invertible in a nbhd of x.

2. (vaguely) allows ~~we~~ us to view (locally) the level sets e.g. $\{x : g(x) = 0\}$ as the graph of a C^1 function as long as $Dg(x)$ ~~invertible~~ non-singular

The following result will be useful for constructing the inverse (and is of independent interest)

Contraction Mapping Thm

Def Let X metric space w/ metric d. If $\varphi: X \to X$ and $\exists$ $c < 1$ s.t.

$$d(\varphi(x), \varphi(y)) \leq c\,d(x,y)$$

then we say φ is a contraction.

Thm If X is complete, and φ is a contraction of X then there exists a unique fixed point of φ, an $x \in X$ s.t. $\varphi(x) = x$.

proof: Uniqueness: if $\varphi(x) = x$ and $\varphi(y) = y$

then $d(\varphi(x), \varphi(y)) = d(x,y) \leq c\,d(x,y)$

$$\Rightarrow \quad d(x,y) = 0.$$

Existence: Pick any $x_0 \in X$ and define x_n by

$$x_{n+1} = \varphi(x_n) \quad \text{for } n = 1, 2, \ldots$$

With c from the contractivity property

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \le c\, d(x_n, x_{n-1})$$

applying inductively

$$d(x_{n+1}, x_n) \le c^n d(x_1, x_0).$$

Then for any pair ~~n<m~~ $m<n$

$$d(x_n, x_m) \le \sum_{i=m}^{n-1} d(x_{i+1}, x_i)$$

$$\le (c^{n-1} + c^{n-2} + \cdots + c^m)\, d(x_1, x_0)$$

$$\le c^m (1 + c + \cdots + c^{n-m-1})\, d(x_1, x_0)$$

$$\le c^m \frac{d(x_1, x_0)}{1-c}$$

This implies that x_n is a Cauchy sequence

since X is complete x_n converges

$$x_n \to x \in X$$

Since φ contraction $\Rightarrow$ (Lipschitz) continuous

$$\varphi(x) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

$\Rightarrow$ x is a fixed point of φ.

<u>Thm</u> (Inverse func theorem) Suppose f is a C' mapping of open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $f'(a)$ is invertible for some $a \in E$ and call $b = f(a)$. Then

(1) $\exists$ open U, V in $\mathbb{R}^n$, $a \in U$, $b \in V$ s.t. f maps U 1-1 onto V

(2) if g is $f^{-1}: V \to U$ then $g \in C'(V)$

and $$g'(y) = f'(f^{-1}(y))^{-1} = f'(g(y))^{-1}$$

<u>Note</u>: We are solving a nonlinear system of n eqns in n unknowns assuming that the "linearized system" can be solved.

$$y_i = f_i(x_1, \dots, x_n) \qquad 1 \le i \le n$$

IFT says this can be solved uniquely for $x(y)$ if we restrict to a small enough nbhd of x, y (as long as we can solve

$$(y - y_0) = f'(x_0)(x - x_0)$$

)

• proof: Call $f'(a) = A$ and let λ so that

$$2\lambda \, \|A^{-1}\| = 1$$

Since $f'(x)$ is cts $\exists$ an open ball $U \ni a$ s.t.

$$\|f'(x) - A\| < \lambda$$

Note that $\|A^{-1}\| \, \|f'(x) - A\| < \lambda \|A^{-1}\| = \frac{1}{2} < 1$

$\Rightarrow$ $f'(x)$ is invertible as well.

Define $\varphi(x) = x + A^{-1}(y - f(x))$

[Note that $\varphi(x) = x$ if and only if $y - f(x) = 0$ i.e. $y = f(x)$]

We show φ is a contraction

$$\varphi'(x) = I - A^{-1} f'(x) = A^{-1}(A - f'(x))$$

so $\|\varphi'(x)\| \leq \|A^{-1}\| \, \|A - f'(x)\| = \frac{1}{2}$

Then since U is a ball (hence convex) Thm says

$\|\varphi(x_1) - \varphi(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\|$ for all $x_1, x_2 \in U$.

Thus φ has ~~exactly~~ at most one fixed point in U.

(φ doesn't map $U \to U$ and also U not complete so we don't know existence yet)

This means that f is 1-1 on U

Call $V = f(U)$ ~~pick $y_0 \in V$ (fixed)~~

We want to show ~~φ has a fixed~~ that V is open as well. Fix $y_0 \in V$ there is a unique $x_0 \in U$ s.t. $f(x_0) = y_0$.

~~Let B an open ball containing centered~~

Let $B = B(x_0, r)$ w/ r small enough that

$$\overline{B} \subset U.$$

Let y s.t. $|y - y_0| < \lambda r$ we want to show $y \in V$

Let $\varphi(x) = x + A^{-1}(y - f(x))$ as before

$$\begin{aligned} |\varphi(x) - x_0| &\leq |\varphi(x) - \varphi(x_0)| + |\varphi(x_0) - x_0| \\ &\leq \frac{1}{2}|x - x_0| + |A^{-1}(y - y_0)| \leq \frac{1}{2}|x - x_0| + \|A^{-1}\| \lambda r \end{aligned}$$

so $x \in B(x_0, r) \Rightarrow$

~~fo $|y - y_0| < \lambda r \Rightarrow$~~

so $|\varphi(x) - x_0| \le \frac{1}{2}|x - x_0| + \frac{1}{2}r \le \frac{1}{2}r + \frac{1}{2}r = r$

i.e. if $x \in \overline{B(x_0, r)}$

$\varphi(x) \in \overline{B(x_0, r)}$

so φ is a contraction of $\overline{B}$ closed subset of $\mathbb{R}^n \Rightarrow$ complete.

Thus φ has a fixed point $x \in \overline{B}$

i.e. $f(x) = y \Rightarrow y \in f(\overline{B}) \subseteq f(U) = V$

~~i.e. $f(\overline{B}$~~

Thus V is open.

<u>proof of (b)</u>: Let $y \in V$, $y + k \in V$

$\exists$ x, $x + h \in U$ s.t.

~~$x = f$~~ $y = f(x)$ $\quad y + k = f(x + h)$

Let φ as before

$$\varphi(x+h) - \varphi(x) = h + A^{-1}[f(x) - f(x+h)] = \cancel{h+A}\ h - A^{-1}k$$

so $|h - A^{-1}k| \leq \frac{1}{2}|h|$ ~~and~~, $|A^{-1}k| \geq \frac{1}{2}|h|$

and so $|h| \leq 2\|A^{-1}\||k| = \lambda^{-1}|k|$

(~~i.e. $|h|$ and $|h|$~~

(i.e. $|h| \to 0$ linearly in $|k|$ as $|k| \to 0$)

Now recalling that $f'(x)$ has an inverse for $x \in V$ call it T

$$g(y+k) - g(y) - Tk = h - Tk = -T[f(x+h) - f(x) - f'(x)h]$$

so

$$\frac{|g(y+k) - g(y) - Tk|}{|k|} \leq \frac{\|T\|}{\lambda} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|}$$

(using $|k| \geq \lambda|h|$ $\to 0$ as $|k| \to 0$ ~~as $|h| \to 0$~~
$(\Rightarrow |h| \to 0)$

Thus g is diff at y and $g'(y) = T = f'(x)^{-1}$
$= f'(g(y))^{-1}$.

Now $g: V \to U$ diff $\Rightarrow$ cts,

f' is a cts mapping $U \to \Omega$

(Ω = {invertible operators in $L(\mathbb{R}^n)$})

and inversion maps $\Omega \to \Omega$ ctsly

So $g'(y) = f'(g(y))^{-1}$

is a composition of cts mappings

so it is continuous.

thus $g \in C'(V)$.

Thm If ~~$f \in C'(E)$~~ $f \in C'(E)$ $f: E \to \mathbb{R}^n$, $E \subset \mathbb{R}^n$

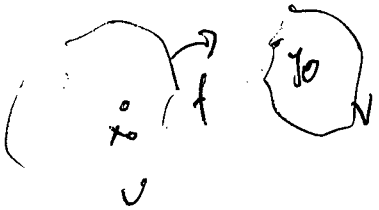
and $f'(x)$ is non-singular for all $x \in E$ then

f is an open mapping $f(W)$ is open $\forall$ W open $\subseteq E$

Alternative proof that $V=f(U)$ is open:

let $y_0\in V$ so $\exists\, x_0\in U$ w/ $y_0=f(x_0)$

let t small s.t. ~~$t=x_0$~~ $B(x_0,t)\subset U$



~~Eqi~~ since f is 1-1 on V

$$|f(x)-y_0|\neq 0 \quad \text{on} \quad \partial B(x_0,t)$$

Call (by compactness) $m=\min\limits_{|x-x_0|=t}\underbrace{|f(x)-y_0|}_{\text{cts fcn of } x}>0$

Let now ~~be~~ y w/ $|y-y_0|<m/3$ and

Call $h(x)=|f(x)-y|^2$ then

for $x\in\partial B(x_0,t)$
$$h(x)=|f(x)-y|^2\geq\left(|f(x)-y_0|-|y-y_0|\right)^2$$
$$\geq(2m/3)^2=4m^2/9>0$$

while $h(x_0)=|y_0-y|^2<m^2/9$

So h must attain its minimum of $\overline{B(x_0,t)}$ at an interior point, $\bar{x}$ there

$$0=D_jh(\bar{x})=2\sum_{k=1}^{m} D_jf_k(\bar{x})\,(f_k(\bar{x})-y_k) \quad \text{for } 1\leq j\leq n$$

i.e. $f'(\bar{x})^T(f(x)-y)=0 \Rightarrow f'(\bar{x})$ not invertible

Third proof that V is open

let $x_1, x_2 \in U$

$$\varphi(x_1) - \varphi(x_2) = x_1 - x_2 + A^{-1}[f(x_2) - f(x_1)]$$

so $$|x_1 - x_2 - A^{-1}(f(x_1) - f(x_2))| \le \tfrac{1}{2}|x_1 - x_2|$$

$$\Rightarrow \quad \tfrac{1}{2}|x_1 - x_2| \le |A^{-1}(f(x_1) - f(x_2))| \le \|A^{-1}\|\,|f(x_1) - f(x_2)|$$

so that $\cancel{|\varphi(x_1) - \varphi}$ $|f(x_1) - f(x_2)| \geq \lambda |x_1 - x_2|$

but this means that $f^{-1}: V \to U$ has the continuity estimate

$$|f^{-1}(y_1) - f^{-1}(y_2)| \le \lambda^{-1}|y_1 - y_2|$$

since this means f^{-1} is cts and

U open, $V = (f^{-1})^{-1}(U) \Rightarrow V$ is open.

$$= \{y : \cancel{y = (f^{-1})^{-1}}\ f^{-1}(y) \in U\}$$