The implicit function theorem

let f a C^1 fn $f: \mathbb{R}^2 \to \mathbb{R}$

a <u>level set</u> of f is $\{ f(x,y) = c \}$ $c \in \mathbb{R}$

consider for example 0-level set

$f(x,y) = 0$ this can be seen as an

implicit equation for y in terms of x
(or x in terms of y)

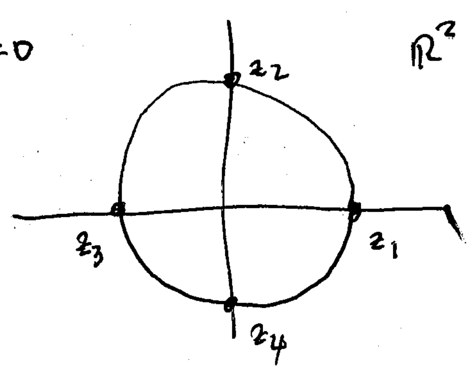
If $f(a,b) = 0$ and $D_1 f(a,b) \neq 0$

the locally we can solve for ~~y(x) s.t.~~ $x(y)$ s.t.

$$f(x(y), y) = 0$$

e.g. consider $f(x,y) = x^2 + y^2 - 1$

$f(x,y) = 0$



$D_2 f(z_1) = D_2 f(z_3) = 0$
so cant solve uniquely for $y(x)$
near z_1, z_3

$D_1 f(z_2) = D_1 f(z_4) = 0$ so cant
solve uniquely for $x(y)$ there

Thm

Linear Version

Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $(x,y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}$

Let $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ we can split A as

$$A_x h = A(h,0) \quad \text{and} \quad A_y k = A(0,k)$$

w/ $A_x \in L(\mathbb{R}^n, \mathbb{R}^n)$, $A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$

Then $A(h,k) = A_x h + A_y k$.

Thm If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and A_x is invertible

Then for each $k \in \mathbb{R}^m$ $\exists!$ $h \in \mathbb{R}^n$ s.t.

$$A(h,k) = 0 \quad \text{i.e.} \quad A(h(k), k) = 0$$

and $h = -A_x^{-1} A_y k$

proof: $0 = A(h,k) = A_x h + A_y k = 0$ iff $h = -A_x^{-1} A_y k$

since A_x is invertible.

Nonlinear Version

Thm (Implicit Fcn Thm) Let f be C' mapping $E \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^n$ such that $f(a,b)=0$ for some $(a,b)\in E$.

Call $A = f'(a,b)$ and assume that A_x is invertible

Then there exists open sets $U \subset \mathbb{R}^{n+m}$, $W \subset \mathbb{R}^m$ w/

$(a,b)\in U$, $b\in W$ ~~having~~ s.t.

for each $y\in W$ $\exists !$ x s.t.

$$(x,y)\in U \quad \text{and} \quad f(x,y)=0$$

If we call this $x = g(y)$ then $g: W \to \mathbb{R}^n$ is C', $g(b)=a$ and

$$f(g(y),y)=0 \quad \text{for } y\in W.$$

and $$g'(b) = -A_x^{-1}A_y$$

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before going into proof lets discuss implications and the relationship w/ <u>inverse function theorem</u>

Implicit Fcn Thm $\Rightarrow$ Inverse fcn thm

let $f : \underset{\text{open}}{E \subseteq \mathbb{R}^n} \to \mathbb{R}^n$ be C' $a \in E$, $f(a) = b$

and $f'(a)$ is non-singular

Then define $h(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$$h(x,y) := f(x) - y \quad , \quad h(a,b) = 0$$

now $\cancel{D_x h = f'}$ $D_x h(a,b) = f'(a)$

is invertible so $\exists$ a nbhd

W of b ~~st.~~ and $g : W \to E$ C' s.t.

$$0 = h(g(y), y) = f(g(y)) - y \quad \text{i.e } g \text{ is inverse of } f.$$

and $g'(b) = -(D_x h)^{-1} D_y h \Big|_{a,b}$

$$= -f'(a)^{-1}(-I) = f'(a)^{-1}.$$

Inverse FT $\Rightarrow$ Implicit FT

let $h : E \subseteq \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h(a,b) = 0$ for

some $(a,b) \in E$

Define $f : E \to \mathbb{R}^n \times \mathbb{R}^m$ by

$$f(x,y) = (h(x,y), y)$$

We want to apply Inverse FT to f.

$$f'(x,y) = \begin{bmatrix} D_x h & D_y h \\ 0 & Id_{m \times m} \end{bmatrix} \quad \text{(block matrix)}$$

$f'(a,b)$ ~~$f'(x,y)$~~ is invertible iff $D_x h \Big|_{(a,b)} \in L(\mathbb{R}^n)$ is invertible.

If $D_x h(a,b)$ invertible then ~~then $\exists$ g on~~

Inverse FT $\Rightarrow$ $\exists$ a ~~A~~ open nbhds

$U \times W$ of (a,b) and $X \times V$ of $(0,b)$
$= (h(a,b), b)$

s.t. $f : U \times W \to X \times V$ 1-1 and onto

Then define $g(y) = \pi_x(f^{-1}(0,y))$

where $\pi_x : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ def by

$\pi_x(x,y) = x$

g defined s.t.

$$f(g(y),y) = (0,y) \quad \text{i.e.} \quad h(g(y),y) = 0$$

Since g is a composition of C' mappings it is C' and differentiating $h(g(y),y) = 0$ we get

$$\cancel{g'(y) = \begin{bmatrix} Id_{n\times n} & 0 \end{bmatrix} f'(g(y),y)^{-1}}$$

$$\cancel{= \begin{bmatrix} Id_{n\times n} & \underbrace{0}_{m} \end{bmatrix} \begin{bmatrix} D_x h\big|_{(g(y),y)} & D_y h\big|_{(g(y),y)} \\ 0 & Id \end{bmatrix}^{-1}}$$

$$\cancel{h'(g(y),y) \begin{bmatrix} g'(y) & 0 \\ 0 & Id \end{bmatrix} = 0}$$

$$0 = h'(g(y),y)\begin{bmatrix} g'(y) \\ I_d \end{bmatrix} = \left[D_x h(g(y),y),\ D_y h(g(y),y)\right]\begin{bmatrix} g'(y) \\ I_d \end{bmatrix}$$

$$= D_x h\, g'(y) + D_y h$$

so $$g'(y) = -(D_x h)^{-1} D_y h \Big|_{(g(y),y)} \qquad \square$$

Applications:

- (Contraction mapping principle) Existence of solutions to systems of ODEs

This will have to wait till we cover integration.