nts

Let $A \in L(\mathbb{R}^n)$, as a matrix

$A = [a_1, \dots, a_n]$ w/ column vectors

$a_j \in \mathbb{R}^n$.

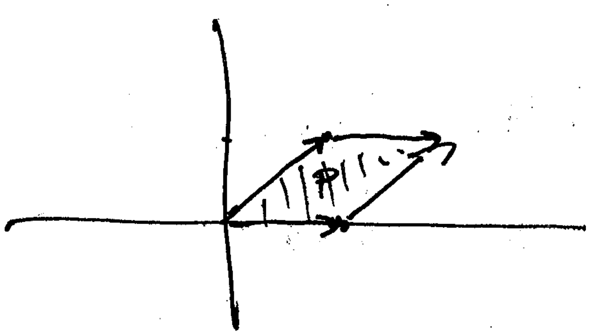
The $\det A$ is the <u>(signed) volume</u>

of the parallelopiped spanned by the

columns.

i.e. $P = \{ x \in \mathbb{R}^n : x = t_1 a_1 + \dots + t_n a_n , \ t_j \in [0,1] \}$

<u>for example</u>: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then



$\text{area}(P) = \text{base} \cdot \text{height}$

Lets give the algebraic defn then see that it is a volume.

Def: If $(j_1, \ldots, j_n)$ is an ordered n-tuple of integers define the signature

$$\sigma(j_1, \ldots, j_n) = \prod_{p<q} \operatorname{sgn}(j_q - j_p)$$

w/

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$$\sigma(j_1, \ldots, j_n) = (-1)^{\# \text{ of inversions}}$$

Let $A \in L(\mathbb{R}^n)$ w/ matrix entries a_{ij} wrt the standard basis.

$$\det A = \sum \sigma(j_1, \ldots, j_n)\, a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

with the sum over all n-tuples $(j_1, \ldots, j_n) \in \{1, \ldots, n\}^n$.

we will think of det as a function of the columns

$\det(a_1, \ldots, a_n) = \det A$

Thm: (1) $\det(I_d) = 1$

(2) det is linear in each column (called multilinear)

i.e. ~~det (A)~~

$\det(\lambda a + b, a_2, \ldots, a_n)$

$= \lambda \det(a, a_2, \ldots, a_n) + \det(b, a_2, \ldots, a_n)$

(3) det is anti-symmetric ~~if~~ in columns

$\det(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = -\det(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n)$

proof: If $A = I$ $a_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$

so $\det A = S(1, 2, \ldots, n) = 1$

Note that $s(j_1, \dots, j_n) = 0$ if any two j_k are equal. Thus each terms of the sum

$$\sum' s(j_1, \dots, j_n)\, a_{j_1} \cdots a_{j_n n}$$

contains exactly one coefficient from each column.

$\Rightarrow$ $\det(a_1, \dots, a_n)$ linear in each entry.

for (3) $s(j_1, \dots, j_n)$ changes sign if two entries are swapped

~~Also note anti-symmetry~~

$\Rightarrow$

~~Also note~~ :

$\det(a_1, \ldots, a_n) = 0$ if two of the columns are the same.

Thm If $A, B \in L(\mathbb{R}^n)$ then

$$\det(\underset{BA}{\cancel{AB}}) = \det(A)\det(B)$$

proof: Call $a_1, \ldots, a_n$ the columns of A

$Ba_1, \ldots, Ba_n$ are the columns of BA

define $\Delta_B(a_1, \ldots, a_n) = \det(BA)$

$$= \det(Ba_1, \ldots, Ba_n)$$

then Δ_B is multi-linear, anti-symmetric

(and if two columns are equal then $\Delta_B = 0$)

We will show $\Delta_B(a_1, \ldots, a_n) = \det(B)\det(A)$

i.e. we are showing that $\det(a_1, \ldots, a_n)$

is the unique multilinear, antisymmetric

functional w/ $\det(I) = 1$.

$$\Delta_B(A) = \Delta_B\left(\textstyle\sum_i' A_{i1} e_1, a_2, \ldots, a_n\right)$$

$$= \sum_i' a_{i1}\, \Delta_B(e_i, a_2, \ldots, a_n)$$

$$= \sum_i' a_{i_1 1}\, a_{i_2 2} \cdots a_{i_n n}\, \Delta_B(e_{i_1}, \ldots, e_{i_n})$$

w/ sum over all n-tuples of $1 \le i_k \le n$.

Of course $\Delta_B(e_{i_1}, \ldots, e_{i_n}) = \sigma(i_1, \ldots, i_n)\, \Delta_B(e_1, \ldots, e_n)$

so $\Delta_B(A) = \sum_i' a_{i_1} \cdots a_{i_n n}\, \sigma(i_1, \ldots, i_n)\, \Delta_B(I)$

$$= \det(A) \det(BI)$$

$$= \det A \det B$$

$\square$

As a corollary the det is actually a function of the linear operator and not the matrix representation

If B is any invertible linear operator

$$\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B)$$
$$= \det(A)$$

since $1 = \det I = \det(B^{-1}B) = \det(B^{-1})\det(B)$.

Thm $|\det(A)|$ is the volume of the parallelopiped spanned by the columns of A.

proof: perform the Gram-Schmidt orthogonalization on the columns of A

$$u_1 = a_1$$
$$u_2 = a_2 - (a_2 \cdot \hat{u}_1)\hat{u}_1$$
$$u_3 = a_3 - (a_3 \cdot \hat{u}_1)\hat{u}_1 - (a_3 \cdot \hat{u}_2)\hat{u}_2$$
$$\vdots$$

$$\det(a_1, \dots, a_n) = \det(u_1, u_2 + q_{12}u_1, \dots, u_n + q_{1n}u_1 + \dots + q_{n-1\,n}u_{n-1})$$
$$= \det(u_1, \dots, u_n)$$
$$= |u_1| \cdots |u_n| \det(\hat{u}_1, \dots, \hat{u}_n)$$

Now since the columns are orthogonal

~~just a~~ with norm one this is

just the identity matrix (up to rotation / inversion)

$\boxed{\begin{array}{l} \text{If } Q \text{ orthogonal} \\ \text{matrix } Q^t Q = Id \\ \text{so } \det(Q^t Q) = 1 \\ \det(Q)^2 = 1 \\ \Rightarrow \det(Q) = \pm 1 \end{array}}$

so $\det(a_1, \dots, a_n) = |u_1| \cdots |u_n|$

Which is the volume. $\square$

Thm: A linear operator $A \in L(\mathbb{R}^n)$ is invertible

iff $\det(A) \neq 0$.

Rem: $\det A \neq 0$ iff the volume of

$A([0,1]^n) = |\det A|$ is non-zero.

Proof: If A invertible we saw $\det(A) = 1$

If A not invertible then ~~$a_1 \in \text{span}(a_2, \dots, a_n)$~~

$\exists \; 1 \leq k \leq n$ s.t.

$a_k = \sum_{j \neq k} c_j a_j \Rightarrow \det A = 0$

using linearity / anti-sym

"The" Jacobian

If $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$

and f is diff at $x \in E$ then we

can define the Jacobian

$$J_f(x) = \det f'(x)$$

or if

$(y_1, \ldots, y_n) = f(x_1, \ldots, x_n)$ we write

$$\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}$$

The Jacobian of f is intrinsically connected w/ the volume change under the transformation f since locally $f(x) = f(x_0) + f'(x_0)(x - x_0) + h.o.t.$ is linear and volume change by linear map is given by determinant.