MATH 275: Homework 1

Due Thursday, April 7

Problem 1. Solve the following initial value / initial boundary value problems using the method of characteristics, you can refer to Chapter 3.1 of the textbook if you need a reference outside of your lecture notes. I usually find it helpful to draw a picture of the characteristics.

(a) Let $f, g : [0, \infty) \to \mathbb{R}$ with $f(0) = g(0)$, solve in terms of f, g:

$$\begin{cases} u_t + x^{1/2} u_x = 0 \ \text{ for } \ (x,t) \in (0,\infty) \times (0,\infty) \\ u(x,0) = f(x) \ \text{ and } \ u(0,t) = g(t) \end{cases}$$

(b) Let $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ write down the general solution of,

$$\begin{cases} u_t + b \cdot \nabla_x u = cu & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R}^n. \end{cases}$$

Problem 2. Use the method of characteristics to solve the following equation

$$\begin{cases} u_t + x|x|u_x = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R}. \end{cases}$$

After finding the general solution, suppose additionally that $f(x) = 0$ for $|x| \leq r$. Show that there is minimal time $T(r)$ so that for any such f the corresponding solution u satisfies $u(\cdot, t) \equiv 0$ for all $t \geq T$, calculate $T(r)$.

Problem 3. Let U any bounded open set of $\mathbb{R}^n$ with smooth (or piecewise smooth) boundary. We call ν to be the outward unit normal to ∂U the boundary of U. Prove the following identities which come up very often in studying Laplace, heat and wave equations:

$$\int_U |\nabla u|^2 + u \Delta u \, dx = \int_{\partial U} u \frac{\partial u}{\partial \nu} \, dS$$

$$\int_U u \Delta v - v \Delta u \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, dS.$$

You can and should make use of the divergence theorem.

Problem 4. [Evans, 1st edition, Ch. 2 Problem 2] Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(Ox) \quad (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

Problem 5. [Evans, 1st edition, Ch. 2 Problem 4] (See Section 8.3 in Shearer and Levy for relevant material) Let U be a bounded domain of $\mathbb{R}^n$. We say $u \in C^2(U)$ is *subharmonic* if

$$-\Delta u \leq 0 \ \text{ in } \ U.$$

(a) Prove for subharmonic v that

$$v(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} v(y) \, dy \ \text{ for all } \ \overline{B(x,r)} \subset U.$$

(b) Prove that the weak maximum principle holds for subharmonic $v \in C^2(U) \cap C(\overline{U})$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v = \phi(u)$. Prove that v is subharmonic.

(d) Prove $v = |Du|^2$ is subharmonic whenever u is subharmonic (you can assume that $|Du|^2$ is $C^2(U)$).

Problem 6. [Bonus]

(a) Let $\rho : \mathbb{R}^n \to [0, \infty)$ be a smooth radially symmetric function supported in $\overline{B(0,1)}$ with $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$ (the existence of such a function is not totally obvious, but suppose you have it). Consider the sequence,

$$\rho_\varepsilon(x) = \varepsilon^{-n} \rho\left(\frac{x}{\varepsilon}\right),$$

this sequence is called a *mollifier*. Suppose that $f \in C(\mathbb{R}^n)$, show that $(\rho_\varepsilon \star f)$ – called a *mollification* of f – is smooth for every $\varepsilon > 0$ and $\rho_\varepsilon \star f \to f$ pointwise and locally uniformly as $\varepsilon \to 0$.

(b) Let $U \subset \mathbb{R}^n$ open, we say that $u \in C(U)$ is *weakly harmonic* if, for every $\varphi \in C_c^2(U)$,

$$\int_U u(x) \Delta \varphi(x) dx = 0.$$

Show that if u is weakly harmonic in U then actually u is harmonic in U as well.