MATH 6410: Ordinary Differential Equations

Bifurcations and Center Manifolds

Instructor: Will Feldman

University of Utah

Bifurcations

Consider a family of ODE systems in $\mathbb{R}^d$

$$\dot{x} = f(x, \varepsilon)$$

defined for each value of the parameter $\varepsilon \in \mathbb{R}^k$.

Definition

We say that ε_0 is a **regular value**, or that a **bifurction does not occur** at ε_0, if there is a neighborhood N of ε_0 so that for all $\varepsilon \in N$ the system $\dot{x} = f(x, \varepsilon)$ is topologically conjugate to $\dot{x} = f(x, \varepsilon_0)$.

Bifurcations

Recall from our study of hyperbolic fixed points and topological conjugacy the following facts:

- A pair of ODE systems with C^1 right hand side are always differentiably conjugate near respective non-fixed points.
- If we have a C^1 parametrized family of ODE systems $\dot{x} = f(x, \varepsilon)$ with a hyperbolic fixed point at $(x_{\varepsilon_0}, \varepsilon_0)$ then there is a continuous family of hyperbolic fixed points $(x_\varepsilon, \varepsilon)$ topologically conjugate to the system $\dot{x} = f(x, \varepsilon_0)$ at $(x_{\varepsilon_0}, \varepsilon_0)$.

This means that the *local* indicator of a bifurcation is a fixed point and a value of ε for which the fixed point is *non-hyperbolic* i.e. it has a non-trivial center subspace/manifold.

Fundamental examples

We will consider some (important) special examples where the center manifold which appears at the bifurcation value is either 1-dimensional because of a single real eigenvalue crossing 0, or 2-dimensional because of a complex conjugate pair passing through the imaginary axis.

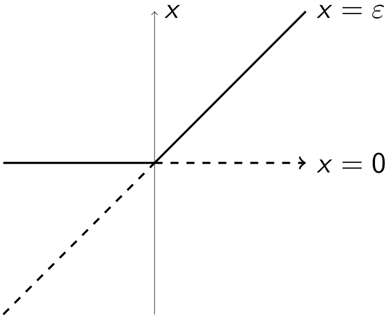
Fundamental examples

Example (Transcritical bifurcation)

Consider the equation

$$\dot{x} = \varepsilon x - x^2.$$

This has fixed points at 0 and ε, one unstable and one stable. When $\varepsilon = 0$ there is a bifurcation, the critical points "collide" and exchange stability.



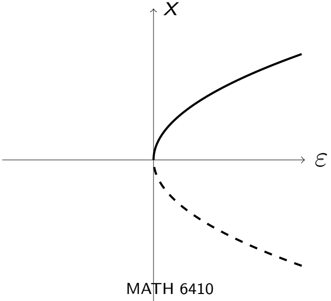
Fundamental examples

Example (saddle-node bifurcation)

Consider the equation

$$\dot{x} = \varepsilon - x^2.$$

This family has a bifurcation at $\varepsilon = 0$. For $\varepsilon < 0$ there are no critical points, at $\varepsilon = 0$ there is a single critical point which is stable on one side and unstable on the other. For $\varepsilon > 0$ there are two critical points at $\pm\sqrt{\varepsilon}$ one stable and the other unstable. The critical points collide at $\varepsilon = 0$ and annihilate.



Fundamental examples

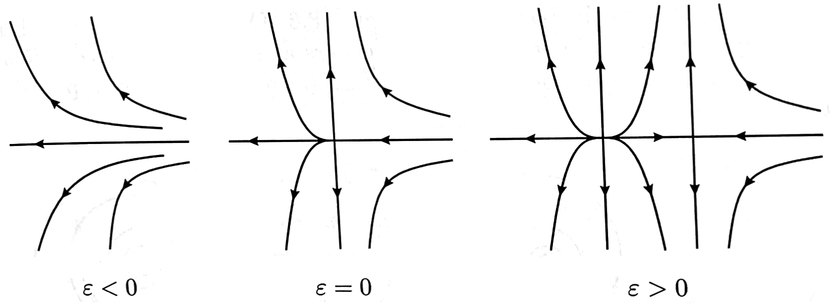
Example (saddle-node bifurcation)

Now consider the two dimensional systems

$$\begin{cases} \dot{x} = \varepsilon - x^2, \\ \dot{y} = y. \end{cases}$$

For $\varepsilon > 0$ this system has a saddle at $(\sqrt{\varepsilon}, 0)$ and an unstable node at $(-\sqrt{\varepsilon}, 0)$. The critical points collide at $\varepsilon = 0$ and annihilate.

Fundamental examples



Ordinary Differential Equations: Qualitative Theory, Luis Barreira and Claudia Valls, Figure 8.4

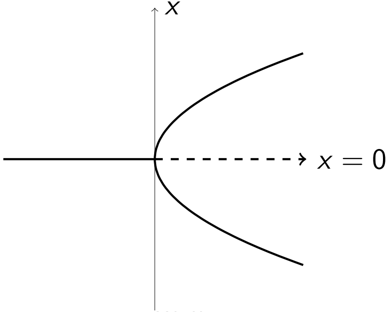
Fundamental examples

Example (pitchfork bifurcation)

Next consider the family of equations

$$\dot{x} = \varepsilon x - x^3.$$

There is only one bifurcation occuring at $\varepsilon = 0$. For $\varepsilon \leq 0$ there is only one critical point at 0, but for $\varepsilon > 0$ there are three critical points at $-\sqrt{\varepsilon}$, $\sqrt{\varepsilon}$ and 0.



Fundamental examples

Example (Hopf bifurcation)

Next consider the family of equations

$$\begin{cases} \dot{x} = \varepsilon x - y - x(x^2 + y^2), \\ \dot{y} = x + \varepsilon y - y(x^2 + y^2). \end{cases}$$

These can be written in polar coordinates

$$\begin{cases} \dot{r} = \varepsilon r - r^3, \\ \dot{\theta} = 1. \end{cases}$$

The r equation has a pitchfork bifurcation at $\varepsilon = 0$, there are two (relevant) stationary solutions for $\varepsilon > 0$ at $r = 0$ and $r = \sqrt{\varepsilon}$, respectively unstable and stable. The stable stationary solution of the r equation at $r = \sqrt{\varepsilon}$ is a stable limit cycle in the (x, y) coordinates.

Hopf Bifurcation

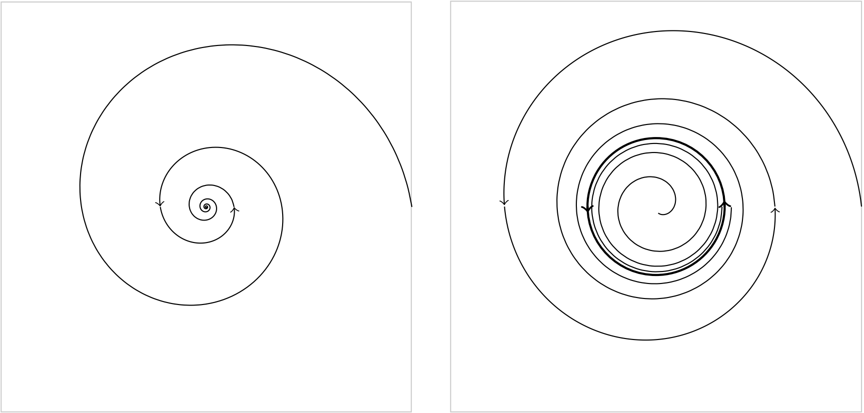


Figure: Hopf bifurcation, stable periodic orbit appears from stable fixed point and fixed point becomes unstable.

Center manifolds

Center manifolds

In general higher dimensional problems all of the “action” in bifurcation theory happens on the center manifold. Recall for an ODE system in $\mathbb{R}^n$

$$\dot{x} = f(x)$$

with a fixed point at 0 and linearization $A = f'(0)$ we have the direct sum decomposition

$$\mathbb{R}^n = E_u \oplus E_s \oplus E_c$$

into the unstable, stable, and center subspaces. For hyperbolic fixed points, when $E_c = \{0\}$, we know there are nonlinear analogues of the stable and unstable manifold.

Center manifold theorem

Theorem

If 0 is a critical point of $\dot{x} = f(x)$ and $f \in C^k$ then there are manifolds $M_s(0)$, $M_u(0)$, and $M_c(0)$ of class C^k containing 0 which are, respectively, tangent to $E_s(0)$, $E_u(0)$, and $E_c(0)$ and invariant under the flow (for small times). The stable and unstable manifold are uniquely determined by these properties (in a small neighborhood of 0).

We have already seen on the homework an example where the center manifold is not unique.

Conjugacy to center manifold dynamics

One can generalize (appropriately) the Hartman-Grobman theorem to show that dynamics near a general non-hyperbolic critical point are conjugate to the following dynamics:

$$\begin{cases} \bar{x}' = -x, \\ \bar{y}' = y, \\ \bar{z}' = F(\bar{z}) \end{cases}$$

where the variables $(\bar{x}, \bar{y}, \bar{z})$ parametrize, respectively, the stable, unstable, and center manifold near the fixed point.

The function F is to be determined, and if E_s or E_u is trivial then the stability/instability of the critical point is entirely dependent on the dynamics by F on the center manifold.

Example of center manifold dynamics

Consider the system

$$\begin{cases} \dot{x} = -x + y^2 \\ \dot{y} = y^2 - x^2. \end{cases}$$

The linearization at 0 is

$$f'(0) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

so the stable subspace is the x-axis and the center subspace is the y-axis.

Example of center manifold dynamics

There is a center manifold $M_c(0)$ which can be, locally, written as a graph over the y-axis $M_c(0) \cap B_\delta(0) = \{(\varphi(y), y) : |y| \leq \delta\}$ with $\varphi(0) = \varphi'(0) = 0$. Thus we have the expansion

$$\varphi(y) = ay^2 + by^3 + \dots$$

up to order k. Now we try to compute the dynamics on the center manifold in terms of the parametrizing variable y.

Example of center manifold dynamics

We substitute in $x = \varphi(y)$ into the dynamics

$$-\varphi(y) + y^2 = \dot{x} = \dot{y}\varphi'(y) = \varphi'(y)(y^2 - \varphi(y)^2)$$

and we plug in the expansion now

$$y^2 - ay^2 - by^3 - \cdots = (2ay + 3by^2 + \cdots)(y^2 - a^2y^4 - \cdots)$$

like we did with asymptotic expansions before we must match terms of each order

$$(1-a)y^2 + (-b-2a)y^3 + O(y^4) = 0$$

resulting in

$$a = 1, \; b + 2a = 0, \quad \text{and higher order conditions.}$$

Example of center manifold dynamics

So we have determined the expansion of the center manifold

$$\varphi(y) = y^2 - 2y^3 + O(y^3)$$

and the expansion of the center manifold dynamics

$$\dot{y} = y^2 - \varphi(y)^2 = y^2 - y^4 + O(y^5)$$

since the origin is unstable for this 1-d ODE we conclude that the origin is unstable for the original system.

Application of center manifold to bifurcation theory

Consider the system

$$\begin{cases} \dot{x} = \varepsilon x - x^3 + y^2 \\ \dot{y} = -y + x^2. \end{cases}$$

we add in the parameter ε as a variable which is independent of t

$$\dot{\varepsilon} = 0.$$

Now the system is in $\mathbb{R}^3$ and there is a center manifold due to the time independent variable ε.

Application of center manifold to bifurcation theory

Now the previous (extended) system is (emphasizing the linearization at the origin

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \varepsilon x - x^3 + y^2 \\ x^2 \\ 0 \end{bmatrix}$$

the stable subspace is the y-axis and the center subspace is the $x\varepsilon$-plane.

Application of center manifold to bifurcation theory

As before let's write the center manifold as a graph over the center subspace

$$M_c(0) = \{(x, \varphi(x, \varepsilon), \varepsilon) : x, \varepsilon \in (-\delta, \delta)\}$$

where $\varphi(0) = 0$ and $\varphi'(0) = 0$.

We plug $y = \varphi(x, \varepsilon)$ into the ODE for y to find the dynamics on the center manifold

$$\dot{y} = \dot{x}\partial_x \varphi + \dot{\varepsilon}\partial_\varepsilon \varphi = (\varepsilon x - x^3 + \varphi^2)\partial_x \varphi$$

and using the equation for y

$$\dot{y} = -y + x^2 = -\varphi + x^2$$

we find the identity

$$(\varepsilon x - x^3 + \varphi^2)\partial_x \varphi = -\varphi + x^2.$$

Application of center manifold to bifurcation theory

Then we plug in the expansion

$$\varphi(x,\varepsilon) = ax^2 + bx\varepsilon + c\varepsilon^2 + \cdots$$

into the identity

$$(\varepsilon x - x^3 + \varphi^2)\partial_x\varphi = -\varphi + x^2$$

to find

$$(\varepsilon x - x^3 + \cdots)(2ax + b\varepsilon + \cdots) = x^2 - ax^2 - bx\varepsilon - c\varepsilon^2 + \cdots$$

and matching terms we find $a = 1$ and $b = c = 0$ so

$$\dot{x} = \varepsilon x - x^3 - \varphi(x,\varepsilon)^2 = \varepsilon x - x^3 + \text{higher order.}$$