MATH 6410: Ordinary Differential Equations

Asymptotic Expansions

Instructor: Will Feldman

University of Utah

Method of Averaging

Preview

We will be turning our focus now from individual systems to parametrized families of systems and investigating the dependence of the properties/features we have already studied on these parameters.

- Solutions (asymptotic expansions).
- Fixed points and periodic orbits.
- Stable and unstable manifolds of hyperbolic fixed points / periodic orbits.

The first example we will discuss is the **method of averaging**.

Fast and slow scales

Consider an n-dimensional system of the form

$$\dot{x} = \varepsilon f(x, t)$$

where f is T-periodic in the t variable and, as usual, C^k.

Since $|\dot{x}| \sim \varepsilon$ the solution x is varying at a **slow** time scale ε^{-1}, while the oscillations of the equation are at a **fast** time scale $T \ll \varepsilon^{-1}$.

The method can also be applied to (weakly) nonlinear oscillators like

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}, t)$$

we will show later how this type of equation can also fit into the averaging framework we will discuss.

Averaged equation

The idea is that the system

$$\dot{x} = \varepsilon f(x, t)$$

is well approximated by an autonomous **averaged system**

$$\dot{a} = \varepsilon \bar{f}(a) \quad \text{with} \quad \bar{f}(y) = \frac{1}{T}\int_0^T f(a, s)\; ds$$

which is the average of the vector field over a unit period in time.

Averaging theorem

Theorem (Method of averaging)

1. *If $x(t)$ and $a(t)$ are respective solutions of the original and averaged equations then*

$$|x(t) - a(t)| \leq C(|x(0) - a(0)| + \varepsilon) \ \text{ for all } \ 0 \leq t \leq L/\varepsilon$$

   where C depends on L and the Lipschitz constant of $\bar{f}$.

2. *If a_0 is a hyperbolic fixed point of the averaged equation then, for all $0 < \varepsilon \leq \varepsilon_0$, there is a unique hyperbolic periodic orbit γ_ε of the original equation with*

$$|\gamma_\varepsilon - a_0| \leq C\varepsilon$$

   and γ_ε has the same stability type of a_0. (Note γ_ε could just be a fixed point).

Averaging theorem

Theorem (Method of averaging continued...)

1. *If* $x_s(t) \in M_s(\gamma_\varepsilon)$ *is a solution of the original equation lying in the stable manifold of a periodic orbit* $|\gamma_\varepsilon - a_0| \leq C\varepsilon$ *and* $a_s(t)$ *is an orbit lying in the stable manifold* $\bar{M}_s(a_0)$ *for the averaged equation with* $|x_s(0) - a_s(0)| \leq C\varepsilon$ *then*

$$|x_s(t) - y_s(t)| \leq \tilde{C}\varepsilon \quad \textit{for} \quad t \in [0, \infty).$$

Motivating asymptotic expansion

If we try to prove this by looking directly at

$$\frac{d}{dt}(x(t)-a(t)) = \varepsilon(f(t,x)-\bar{f}(a)) = \varepsilon(\bar{f}(x)-\bar{f}(a))+\varepsilon(f(t,x)-\bar{f}(x))$$

we are stuck because the best we can say is $f(t,x) - \bar{f}(x) = O(1)$ so we could obtain an error estimate (by Grönwall) like

$$|x(t) - a(t)| \leq e^{\varepsilon\|D\bar{f}\|_{sup}t}(|x(0) - a(0)| + \varepsilon t),$$

which by time ε^{-1} is a unit size error.

We need to introduce an additional correction term to our approximation scheme.

Asymptotic expansion

We want to separate out the slow scale and fast scale dependence so we make an ansatz

$$x(t) = a(t) + \varepsilon b(a(t), t) + O(\varepsilon^2)$$

with $\omega(a, t)$ a T-periodic function of t and mean zero

$$\int_0^T b(a, t)\ dt = 0.$$

Asymptotic expansion

The slow time scale equation is

$$\dot{a} = \varepsilon \bar{f}(a) \quad \text{where } \bar{f}(a) = \frac{1}{T}\int_0^T f(a,s)\, ds$$

with $a(0) = x(0)$ and the fast time scale equation is

$$\dot{b}(t,a) = (f(a,t) - \bar{f}(a)) \quad \text{with} \quad b(a,0) = 0.$$

The fast time scale equation has an explicit solution

$$b(t,a) = \int_0^t f(a,s) - \bar{f}(a)\, ds$$

which is indeed T-periodic.

Formal derivation of the asymptotic expansion

To explain how one would come up with this let's start with a more abstract asymptotic expansion of x in terms of functions varying at the t and $\tau = \varepsilon t$ time scales

$$x(t,\tau) = x_0(t,\tau) + \varepsilon x_1(t,\tau) + \cdots .$$

Note that the operator $\frac{d}{dt}$ acts on functions $x_j(t, \varepsilon t)$ by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}.$$

Formal derivation of the asymptotic expansion

We plug in the previous expansion into the ODE paying attention to terms of order 1 and of order ε

$$\begin{aligned} 0 &= \frac{d}{dt}x(t,\tau) - \varepsilon f(x,t) \\ &= \frac{\partial x_0}{\partial t} + \varepsilon \left[\frac{\partial x_0}{\partial \tau} + \frac{\partial x_1}{\partial t} - f(x_0,t) \right] + O(\varepsilon^2). \end{aligned}$$

Since the expansion is meant to be valid for all $\varepsilon > 0$ we can send $\varepsilon \to 0$ and find that the unit order term must be zero

$$\frac{\partial x_0}{\partial t}(t,\tau) = 0 \implies x_0(t,\tau) = x_0(\tau).$$

The first term in the expansion depends only on the slow time scale τ.

Formal derivation of the asymptotic expansion

Now the asymptotic expansion says

$$0 = \varepsilon \left[\frac{\partial x_0}{\partial \tau} + \frac{\partial x_1}{\partial t} - f(x_0, t) \right] + O(\varepsilon^2)$$

we can divide the whole expression by ε and send $\varepsilon \to 0$ to find that

$$\frac{\partial x_1}{\partial t} = f(x_0, t) - \frac{\partial x_0}{\partial \tau}(\tau)$$

we may as well put $x_1(0, \tau) = 0$ so

$$x_1(t, \tau) = \int_0^t f(x_0(\tau), s) - \frac{\partial x_0}{\partial \tau}(\tau)\ ds.$$

We need $x_1(t, \tau)$ to be bounded in t for each τ since otherwise the asymptotic expansion will be invalid by time ε^{-1}. Since the right hand side is T-periodic this requires

$$\int_0^T f(x_0(\tau), s) - \frac{\partial x_0}{\partial \tau}(\tau)\ ds = 0 \quad \text{or} \quad \frac{\partial x_0}{\partial \tau} = \bar{f}(x_0(\tau)).$$

Error estimate of the asymptotic expansion

Now we would like to compare asymptotic expansion

$$y(t) = a(t) + \varepsilon b(a(t), t)$$

with the true solution $x(t)$ of

$$\dot{x} = \varepsilon f(x, t).$$

Lemma

Suppose that $f(t, x)$ is Lipschitz continuous in x uniformly in t. If y and x are as above then for $t \in [0, L\varepsilon^{-1}]$ there is a constant C depending on L and $\|Df\|_{sup}$ so that

$$|x(t) - y(t)| \leq C(|x(0) - y(0)| + \varepsilon^2 t)$$

Error estimate

The error estimate will come from plugging $y(t)$ into the equation for x, estimating how far it is from being a solution, and applying Grönwall.

$$\begin{aligned}
\dot{y} &= \dot{a}(t) + \varepsilon(\dot{b}(a(t),t) + b_a(a(t),t)\dot{a}(t)) \\
&= \varepsilon\bar{f}(a) + \varepsilon(f(a,t) - \bar{f}(a)) + \varepsilon^2 b_a(a,t)\bar{f}(a) \\
&= \varepsilon f(a,t) + \varepsilon^2 b_a(a,t)\bar{f}(a) \\
&= \varepsilon f(y,t) + \varepsilon^2[b_a(a,t)\bar{f}(a) + \varepsilon^{-1}(f(a+\varepsilon b,t) - f(a,t))].
\end{aligned}$$

Note that the Lipschitz assumption on f implies

$$\varepsilon^{-1}|f(a+\varepsilon b,t) - f(a,t)| \leq \|Df\|_{sup}|b(a,t)|$$

and let's just call C to be an upper bound on the term in square brackets above.

Error estimate

So we can apply Grönwall to

$$\frac{d}{dt}|x-y| \le \varepsilon \|Df\|_{sup}|x-y| + C\varepsilon^2$$

to find

$$|x(t)-y(t)| \le e^{\varepsilon\|Df\|_{sup}t}|x(0)-y(0)|+e^{\varepsilon\|Df\|_{sup}t}\int_0^t e^{-\varepsilon\|Df\|_{sup}s}C\varepsilon^2\,ds$$

which is

$$|x(t)-y(t)| \le e^{\varepsilon\|Df\|_{sup}t}(|x(0)-y(0)|+C\varepsilon^2 t).$$

Hyperbolic stationary points perturb to periodic orbits

To find periodic orbits of the non-averaged system we will compare the Poincaré maps P_0 and P_ε for the respective equations associated with the transversal manifold $\Sigma = \mathbb{R}^n \times \{t = 0\}$ and the period T of the non-averaged system, we already know

$$|P_\varepsilon(x) - P_0(x)| \leq C\varepsilon^2.$$

If x_0 is a hyperbolic stationary point for $\bar{f}$ then

$$DP_0(x_0) = e^{\varepsilon T D\bar{f}(x_0)}$$

is also a hyperbolic multiplier matrix.

Hyperbolic stationary points perturb to periodic orbits

Recall that if $P_\varepsilon(x_\varepsilon) = x_\varepsilon$ then the solution with initial data x_ε is a periodic orbit with period T. So we are looking for a solution curve x_ε of the implicit equation

$$P_\varepsilon(x) - x = 0$$

with $x_{\varepsilon=0} = x_0$.

To apply implicit function theorem we should check

$$DP_\varepsilon(x_0) - I \quad \text{is invertible for small } \varepsilon > 0$$

Computation

Let's assume $x_0 = 0$ to simply computations then we are trying to solve

$$\begin{aligned} 0 = P_\varepsilon(h) - h &= P_\varepsilon(h) - P_0(h) + P_0(h) - h \\ &= O(\varepsilon^2) + P_0(h) - e^{\varepsilon D\bar{f}(x_0)T} h + (e^{\varepsilon D\bar{f}(x_0)T} - I)h \\ &= (e^{\varepsilon D\bar{f}(x_0)T} - I)h + O(\varepsilon^2) + O(\varepsilon|h|^2) \\ &= \varepsilon D\bar{f}(x_0)Th + O(\varepsilon^2) + O(\varepsilon|h|^2) \end{aligned}$$

since the matrix $D\bar{f}(x_0)$ is invertible by assumption we can find a value of $h = O(\varepsilon)$ so that $P_\varepsilon(h) - h = 0$.

Linear example

For example consider the scalar equation

$$\dot{x} = \varepsilon x \sin^2 t$$

the right hand side has period π. The averaged flow field is

$$\bar{f}(a) = \frac{1}{\pi} \int_0^{\pi} a \sin^2 t \, dt = \frac{1}{2} a.$$

Linear example

So the averaged equation is

$$\dot{a} = \frac{\varepsilon}{2} a$$

and the first order correction is

$$b(t, a) = \int_0^t a \sin^2 s - \frac{1}{2} a \ ds = -a\frac{1}{2} \sin(t) \cos(t) = -a\frac{1}{4} \sin(2t)$$

so our expansion says

$$x(t) = a(t)(1 - \frac{\varepsilon}{4} \sin(2t)) + O(\varepsilon^3 t) \ \text{ with } \ a(t) = e^{\varepsilon t/2} x(0)$$

for $t \in [0, L\varepsilon^{-1}]$.

Weakly nonlinear oscillators

A common situation where one wants to apply the method of averaging is a nonlinear oscillator with natural (linear) frequency ω_0,

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}, t)$$

and nonlinearity f which is ω^{-1}-periodic with $\omega \approx k\omega_0$ (i.e. the period $T = \omega^{-1} = \frac{1}{k} T_0$ where $T_0 = \omega_0^{-1}$ is the natural period).

This equation can be transformed into a form amenable to the method of averaging.

Homogeneous solution

Recall that the homogeneous part of the equation

$$\ddot{x} + \omega_0^2 x = 0$$

can be solved by

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = e^{\Omega_0 t} \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix}$$

where

$$\Omega_0 = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \quad \text{and} \quad e^{\Omega_0 t} = \begin{bmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{bmatrix}$$

Aside about resonances

Since we did not study this carefully before let me also make some mention about the phenomenon of **resonance** as relates to forced oscillators

$$\ddot{x} + \omega_0^2 x = f(t)$$

with $f(t)$ being T-periodic. The integer relation (or not) between the period T and the natural period $T_0 = \omega_0^{-1}$ is central.

It turns out that if the ratio of T/T_0 is not rational then all solutions are bounded and **quasi-periodic** while if T/T_0 is rational then the system *may* be **resonant** depending on f having linearly growing solutions. If the system is non-resonant and T/T_0 is rational then every initial data gives a periodic (bounded) solution.

Resonance

By Duhamel

$$x(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = e^{\Omega_0 t} \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} + e^{\Omega_0 t} \int_0^t \begin{bmatrix} -\sin(\omega_0 s) f(s) \\ \cos(\omega_0 s) f(s) \end{bmatrix} \, ds$$

if T and T_0 have least common integer multiple $S = kT = \ell T_0$ then the integral term above can be written as

$$\lfloor t/S \rfloor \int_0^S \begin{bmatrix} -\sin(\omega_0 s) f(s) \\ \cos(\omega_0 s) f(s) \end{bmatrix} \, ds + \int_{\lfloor t/S \rfloor S}^t \begin{bmatrix} -\sin(\omega_0 s) f(s) \\ \cos(\omega_0 s) f(s) \end{bmatrix} \, ds.$$

The second integral is over an interval of length at most S and hence is bounded independent of t, but the first term will grow linearly in t unless $f(s)$ is **orthogonal** to both $\sin(\omega_0 s)$ and $\cos(\omega_0 s)$.

Nonlinear oscillators

Now let's return to the nonlinear oscillator

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}, t).$$

In general we will need the method of multiple scales / method of averaging to compute an expansion for this type of problem, but let's start with a special case where a simpler asymptotic expansion approach works.

Regular perturbation series

van der Pol oscillator

Let's look at the van der Pol oscillator

$$\ddot{x} + \varepsilon \dot{x}(x^2 - 1) + x = 0$$

and start with a **regular perturbation series** ansatz

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots.$$

Regular perturbation series

van der Pol oscillator

Plugging in our ansatz to the equation we find

$$[\ddot{x}_0 + x_0] + \varepsilon[\ddot{x}_1 + x_1 + \dot{x}_0(x_0^2 - 1)] + O(\varepsilon^2) = 0$$

as before we first set the highest order term equal to zero

$$\ddot{x}_0 + x_0 = 0$$

which implies

$$x_0(t) = R_0 \cos(t + \theta_0).$$

Let's simplify computations and focus on initial data with $\theta_0 = 0$.

Regular perturbation series

van der Pol oscillator

Moving to the next term in the series we find

$$\ddot{x}_1 + x_1 = -\dot{x}_0(x_0^2 - 1)$$

or

$$\ddot{x}_1 + x_1 = R_0 \sin(t)(R_0^2 \cos(t) - 1)$$

the right hand side can be simplified using trigonometric identities to

$$\ddot{x}_1 + x_1 = \frac{1}{4} R_0 (R_0^2 - 4) \sin(t) + \frac{1}{4} R_0^3 \sin(3t)$$

The first term is resonant so we eliminate it by choosing

$$R_0 = 2.$$

Regular perturbation series

van der Pol oscillator

This results in an asymptotic expansion

$$x(t) = 2\cos(t) + O(\varepsilon)$$

which is valid for times up to $O(\varepsilon^{-1})$.

However we had to choose $R_0 = 2$! This fixes a single choice of initial data for which our series expansion works. The problem is that other choices of R_0 are evolve at a slow time scale due to the perturbative term, we need a better choice of asymptotic expansion!

Regular perturbation series

Duffing equation

Now let's see an example where the regular perturbation series approach does not work at all, the **Duffing equation**:

$$\ddot{x} + x = -\varepsilon x^3$$

We will try the same ansatz

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots.$$

Regular perturbation series

Duffing equation

Plugging in our ansatz to the equation we find

$$[\ddot{x}_0 + x_0] + \varepsilon[\ddot{x}_1 + x_1 + x_0^3] + O(\varepsilon^2) = 0$$

as before we first set the highest order term equal to zero

$$\ddot{x}_0 + x_0 = 0$$

which implies

$$x_0(t) = R_0 \sin(t + \theta_0).$$

Let's simplify computations and focus on initial data with $\theta_0 = 0$.

Regular perturbation series

Duffing equation

Moving to the next term in the series we find

$$\ddot{x}_1 + x_1 = -x_0^3$$

or

$$\ddot{x}_1 + x_1 = -R_0^3 \sin^3(t)$$

the right hand side can be simplified using trigonometric identities to

$$\ddot{x}_1 + x_1 = \frac{1}{4} R_0^3 (\sin(3t) - 3\sin(t))$$

The second term is resonant but there is no way to choose R_0 to cancel!

The problem is that the nonlinear perturbation slightly changes the period of the system and our asymptotic expansion could not capture this!

Direct approach

We can resolve these issues in general for

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}, t)$$

by making a better perturbation series using the **method of multiple scales**.

As motivation note that solutions of the homogenous equation have the form

$$x_h(t) = r_0 \cos(\omega_0 t + \theta_0)$$

if we believe that the magnitude and frequency are also evolving at a slow time scale due to the perturbation we could propose an ansatz of the form

$$x(t) = R(\varepsilon t) \sin(\omega_0 t + \Theta(\varepsilon t)) + O(\varepsilon)$$

where the amplitude and phase are varying at the slow time scale $\tau = \varepsilon^{-1} t$. We will need to find equations for $R(\tau)$ and $\Theta(\tau)$.

Direct approach

Let's derive this form by the asymptotic expansion method from earlier

$$x(t,\tau) = x_0(t,\tau) + \varepsilon x_1(t,\tau) + \cdots$$

where, again, $\tau = \varepsilon t$ is the slow time scale.

Note that the operator $\frac{d}{dt}$ acts on functions $x_j(t, \varepsilon t)$ by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}$$

and

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial}{\partial \tau^2}.$$

Direct approach

We plug into the ODE and find

$$\begin{aligned} 0 &= \ddot{x} + x - \varepsilon f(x, \dot{x}, t) \\ &= \frac{\partial^2 x_0}{\partial t^2} + \omega_0^2 x_0 + \varepsilon\Big(2\frac{\partial^2 x_0}{\partial t \partial \tau} + \frac{\partial^2 x_1}{\partial t^2} + \omega_0^2 x_1 - \varepsilon f(x_0, \dot{x}_0, t)\Big) + O(\varepsilon^2). \end{aligned}$$

We set equal to zero the coefficient in front of each power of ε starting with the lowest power

$$\frac{\partial^2 x_0}{\partial t^2} + \omega_0^2 x_0 = 0$$

i.e.

$$x_0(t, \tau) = R(\tau) \sin(\omega_0 t + \Theta(\tau))$$

which justifies the ansatz we had proposed earlier.

Direct approach

Next we consider the $O(\varepsilon)$ coefficient and set this equal to zero as well

$$\frac{\partial^2 x_1}{\partial t^2} + \omega_0^2 x_1 = -2\frac{\partial^2 x_0}{\partial t \partial \tau} + f(x_0, \partial_t x_0, t)$$

using the form of $x_0(t, \tau)$ we can write

$$\begin{aligned}\frac{\partial^2 x_1}{\partial t^2} &+ \omega_0^2 x_1 \\ &= 2\omega_0[-R'(\tau)\cos(\omega_0 t + \Theta) + R\Theta'(\tau)\sin(\omega_0 t + \Theta)] + f(x_0, \partial_t x_0, t)\end{aligned}$$

To go further we will need to fix an explicit choice of f, however the goal here is to choose R and Θ so that there is a solution x_1 of this equation which is periodic in t. This means eliminating any resonant terms on the right hand side.

Eliminating resonant terms

We take the L^2 inner product on $[0, S]$ of the right hand side with $\sin(\omega_0 t + \Theta)$ and $\cos(\omega_0 t + \Theta)$ and set both equal to zero. By orthogonality of sin and cos this gives two conditions

$$\frac{1}{S}\int_0^S -2\omega_0 R'(\tau)\cos^2(\omega_0 t+\Theta)+\cos(\omega_0 t+\Theta)f(x_0,\partial_t x_0,t)\;dt = 0$$

and

$$\frac{1}{S}\int_0^S 2\omega_0 R\Theta'(\tau)\sin^2(\omega_0 t+\Theta)+\sin(\omega_0 t+\Theta)f(x_0,\partial_t x_0,t)\;dt = 0.$$

These simplify a bit:

Eliminating resonant terms

Simplified conditions

$$\omega_0 R'(\tau) = \frac{1}{S}\int_0^S \cos(\omega_0 t + \Theta) f(x_0, \partial_t x_0, t)\ dt$$

and

$$\omega_0 R\Theta'(\tau) = -\frac{1}{S}\int_0^S \sin(\omega_0 t + \Theta) f(x_0, \partial_t x_0, t)\ dt.$$

These will give ODE for R and Θ which can be solved to complete our computation of the first term in the asymptotic expansion. The point is not to remember these precise expressions but to remember the method of derivation.

Duffing equation

Let's consider the **Duffing equation** again

$$\ddot{x} + x = -\varepsilon x^3.$$

In the multiple scales framework we have the expansion

$$x(t) = R(\varepsilon t)\sin(t + \Theta(\varepsilon t)) + O(\varepsilon)$$

where R and Θ are determined by the non-resonance condition at the ε order term when the asymptotic expansion is plugged into the ODE.

Duffing equation expansion

Recall that the ε order term in the asymptotic expansion was

$$\frac{\partial^2 x_1}{\partial t^2} + \omega_0^2 x_1 = -2\frac{\partial^2 x_0}{\partial t \partial \tau} - x_0^3$$

where we can write the RHS more explicitly

$$2[-R'(\tau)\cos(t+\Theta) + R\Theta'(\tau)\sin(t+\Theta)] - R^3\sin^3(t+\Theta).$$

We can use the identity

$$\sin^3(t+\Theta) = \frac{1}{4}[3\sin(t+\Theta) - \sin(3(t+\Theta))].$$

The second term here is non-resonant, it is orthogonal to both sin and cos on L^2 of $[0, 2\pi]$.

Duffing equation expansion

If we just collect the two resonant terms and set the coefficients equal to zero we find

$$R'(\tau) = 0 \quad \text{and} \quad 2R\Theta'(\tau) = \frac{3}{4}R^3$$

which can be solved to find

$$R(\tau) = R_0 \quad \text{and} \quad \Theta(\tau) = \Theta_0 + \frac{3}{8}R_0^2\tau.$$

Duffing equation asymptotic expansion

In particular we derived the following expansion, which is valid for $t \leq O(\varepsilon^{-1})$,

$$x(t) = R_0 \sin(t + \Theta_0 + \frac{3}{8} R_0^2 \varepsilon t) + O(\varepsilon)$$

of the solution of Duffing's equation

$$\ddot{x} + x = -\varepsilon x^3.$$

Note: this particular system is planar Hamiltonian and we could find exact solutions and prove that all solutions are periodic in that way.

Connection with method of averaging

There is obviously some similarity between the method of multiple scales approach we have taken for the equation

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}, t)$$

and the method of averaging we saw for first order systems

$$\ddot{y} = \varepsilon F(y, t).$$

We can actually make a change of variables in the nonlinear oscillator to see the multiple scales expansion as a special case of the averaging method.

Undoing the homogeneous part

Similar to Duhamel we make a change of variables to undo the homogeneous evolution

$$Y(t) = e^{-\Omega_0 t} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = e^{-\Omega_0 t} X(t)$$

where $X(t)$ solves

$$\dot{X} = \Omega_0 X + \varepsilon F(X,t) \text{ with } F(X,t) = \begin{bmatrix} 0 \\ f(X_1, X_2, t) \end{bmatrix}$$

So that

$$\begin{aligned} \dot{Y} &= -\Omega_0 Y + e^{\Omega_0 t} \dot{X} \\ &= -\Omega_0 Y + e^{-\Omega_0 t}(\Omega_0 X + \varepsilon F(X,t)) \\ &= \varepsilon e^{-\Omega_0 t} F(X,t) = \varepsilon e^{-\Omega_0 t} F(e^{\Omega_0 t} Y, t). \end{aligned}$$

Transformed equation

Now the ODE is in the standard form for the method of averaging

$$\dot{Y} = \varepsilon e^{-\Omega_0 t} F(e^{\Omega_0 t} Y, t)$$

and the right hand side is S periodic where S is the least common multiple of $T_0 = \omega_0^{-1}$ and T (the period of f).

The asymptotic expansion from before will be

$$Y(t) = A(\varepsilon t) + \varepsilon B(A(\varepsilon t), t) + O(\varepsilon^2)$$

where

$$\dot{A} = \bar{F}(A) = \frac{1}{S} \int_0^S e^{-\Omega_0 t} F(e^{\Omega_0 t} A, t)\, dt$$

Corollaries of the averaging theorem

Our arguments for the nonlinear oscillator

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}, t)$$

have so far been formal, but given the connection with the method of averaging where we (partially) proved a rigorous result we can conclude:

- If $R(\tau)$ and $\Theta(\tau)$ solve the appropriate averaged equations with initial data (R_0, Θ_0) then, for any $L > 1$

  $$|x(t) - R(\varepsilon t)\cos(t + \Theta(\varepsilon t))| \leq C(L)\varepsilon \quad \text{for} \quad 0 \leq t \leq L\varepsilon^{-1}.$$

  (This uses a Lipschitz bound on $f(x, \dot{x}, t)$ with respect to $(x, \dot{x})$ variables, non-Lipschitz nonlinearities are OK as long as we have some way to guarantee that our solution stays inside some compact set)

Poincaré-Linstedt Method

The connection with method of averaging does not directly give us a way to show persistence of periodic orbits for a perturbed oscillator

$$\ddot{x} + x = \varepsilon f(x, \dot{x})$$

there is no stationary solution of the averaged equations because the period is changing in ε.

Let's also expand the frequency

$$\omega^{-2} = 1 - \varepsilon \eta(\varepsilon)$$

and look in the new variable $\tau = \omega t$

$$x(t) = y(\omega t).$$

Equation in new variables

In the new variable τ the equation becomes

$$\omega^2 y'' + y = \varepsilon f(y, \omega y')$$

or

$$y'' + y = (1 - \omega^{-2})y + \varepsilon\omega^{-2} f(y, \omega y')$$
$$y'' + y = \varepsilon\eta(\varepsilon)y + \varepsilon\omega^{-2} f(y, \omega y')$$
$$y'' + y = \varepsilon F(y, y', \eta(\varepsilon), \varepsilon)$$

Duhamel

Now we integrate the equation using Duhamel

$$y(\tau) = a\cos\tau + \varepsilon \int_0^\tau F(y(s), y'(s), \eta(\varepsilon), \varepsilon)[\sin\tau\cos s - \cos\tau\sin s]ds$$

the condition for 2π-periodicity is

$$\int_\tau^{\tau+2\pi} F(y(s), y'(s), \eta(\varepsilon), \varepsilon)[\sin\tau\cos s - \cos\tau\sin s]ds = 0$$

for all τ. Since $\sin\tau$ and $\cos\tau$ are linearly independent this means that the coefficients of both must be zero.

Periodicity conditions

In other words there is a pair of conditions for periodicity

$$\int_0^{2\pi} F(y(s), y'(s), \eta(\varepsilon), \varepsilon) \cos s \; ds = 0$$
$$\int_0^{2\pi} F(y(s), y'(s), \eta(\varepsilon), \varepsilon) \sin s \; ds = 0$$

Let's define

$$I_1(a, \eta, \varepsilon) := \int_0^{2\pi} F(y(s), y'(s), \eta, \varepsilon) \cos s \; ds$$
$$I_2(a, \eta, \varepsilon) := \int_0^{2\pi} F(y(s), y'(s), \eta, \varepsilon) \sin s \; ds$$

where $y(s, a, \eta, \varepsilon)$ is the solution of

$$y'' + y = \varepsilon F(y, y', \eta, \varepsilon) = \varepsilon[\eta y + (1 - \eta\varepsilon) f(y, (1 - \eta\varepsilon)^{-1/2} y')]$$

with initial data $y(0) = a$ and $y'(0) = 0$.

Periodicity conditions

The periodicity conditions are then

$$I_1(a, \eta, \varepsilon) = 0$$
$$I_2(a, \eta, \varepsilon) = 0.$$

The idea is to use the implicit function theorem, if

$$\left| \frac{\partial(I_1, I_2)}{\partial(a, \eta)} \right| (a_0, \eta_0, 0) \neq 0$$

for an appropriate choice of (a_0, η_0) then there will be a solution curve

$$I_j(a(\varepsilon), \eta(\varepsilon), \varepsilon) = 0 \ \text{ for } \ j = 1, 2 \ \text{ and } \varepsilon > 0 \text{ small.}$$

Computing the IFT condition

We need to compute at $\varepsilon = 0$ which majorly simplifies

$$I_1(a, \eta, 0) = \int_0^{2\pi} [\eta a \cos(s) + f(a\cos(s), -a\sin(s))] \cos s \, ds$$

or

$$I_1(a, \eta, 0) = \pi a \eta + \int_0^{2\pi} f(a\cos(s), -a\sin(s)) \cos s \, ds$$

and we need to compute the derivatives

$$\frac{\partial I_1}{\partial \eta}(a, \eta, 0) = \pi a$$

and

$$\frac{\partial I_1}{\partial a}(a, \eta, 0) = \pi \eta + \int_0^{2\pi} [\frac{\partial f}{\partial x} \cos(s) - \frac{\partial f}{\partial \dot{x}} \sin(s)] \cos s \, ds$$

Computing the IFT condition

For I_2 the computations are similar

$$I_2(a,\eta,0) = \int_0^{2\pi} [\eta a \cos(s) + f(a\cos(s), -a\sin(s))] \sin s \; ds$$

or

$$I_2(a,\eta,0) = \int_0^{2\pi} f(a\cos(s), -a\sin(s)) \sin s \; ds$$

and the derivatives are

$$\frac{\partial I_2}{\partial \eta}(a,\eta,0) = 0$$

and

$$\frac{\partial I_2}{\partial a}(a,\eta,0) = \int_0^{2\pi} [\frac{\partial f}{\partial x}\cos(s) - \frac{\partial f}{\partial \dot{x}}\sin(s)] \sin s \; ds.$$

Computing the IFT condition

Finally we combine into the condition

$$\left|\frac{\partial(I_1, I_2)}{\partial(a,\eta)}\right|_{(a,\eta,0)} = \left[\frac{\partial I_1}{\partial a}\frac{\partial I_2}{\partial \eta} - \frac{\partial I_1}{\partial \eta}\frac{\partial I_2}{\partial a}\right]_{(a,\eta,0)}$$
$$= \pi a \int_0^{2\pi} [\frac{\partial f}{\partial x}\cos(s) - \frac{\partial f}{\partial \dot{x}}\sin(s)]\sin s \ ds \neq 0$$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial \dot{x}}$ in the equation above are evaluated at the point $(a\cos(s), -a\sin(s))$.

Application of Poincaré-Linstedt

Let's consider the application of Poincaré-Linstedt to to the van der Pol oscillator

$$\ddot{x} + x = \varepsilon \dot{x}(1 - x^2).$$

We first need to find an appropriate starting point $(a_0, \eta_0, 0)$ with

$$I_j(a_0, \eta_0, 0) = 0.$$

From the method of multiple scales we already know that $a_0 = 2$ and $\eta_0 = 0$, but let's see anyway how these will be fixed by the equations we have derived with the Poincaré-Linstedt method.

Application of Poincaré-Linstedt to van der Pol equation

The first condition is

$$0 = I_1(a, \eta, 0) = \pi a\eta + \int_0^{2\pi} (-a \sin s)(1 - a^2 \cos^2 s) \cos s \; ds = \pi a\eta$$

which gives

$$a\eta = 0 \;\text{ which will mean }\; \eta = 0.$$

The second condition is

$$0 = I_2(a, \eta, 0) = \int_0^{2\pi} (-a \sin s)(1-a^2 \cos^2 s)(\sin s) \; ds = -a\pi + a^3 \frac{\pi}{4}$$

which gives

$$a^2 = 4 \;\text{ or }\; a = 2$$

so indeed we need to choose $\eta = 0$ to satisfy the first equation.

Application of Poincaré-Linstedt to van der Pol equation

Finally we need to check the continuation condition

$$0 \neq \left| \frac{\partial(I_1, I_2)}{\partial(a, \eta)} \right|_{(2,0,0)}$$

which is

$$0 \neq \pi a \int_0^{2\pi} [2a^2 \sin s \cos^2 s - (1 - a^2 \cos^2 s) \sin s] \sin s \; ds.$$