MATH 6410: Ordinary Differential Equations

Nonlinear analysis near hyperbolic points/orbits

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Stable manifold theorem

Stable and unstable subspaces

Let's recall what we already know about linear systems

$$\dot{x} = Ax.$$

There are **stable** and **unstable subspaces**

$$E_s(A) = \bigoplus_{\mathrm{Re}(\lambda_j)<0} \mathrm{Ker}(A-\lambda)^{m_j} \quad \text{and} \quad E_u(A) = \bigoplus_{\mathrm{Re}(\lambda_j)>0} \mathrm{Ker}(A-\lambda)^{m_j}$$

where, if $x_0 \in E_s$ then $x(t) \to 0$ as $t \to \infty$ and if $x_0 \in E_u$ then $x(t) \to 0$ as $t \to -\infty$.

There is also the center manifold E_c, but without special structural hypotheses this will be unstable under the perturbation to nonlinearity. To avoid this issue we can consider **hyperbolic** systems which have no center subspace.

Global stable and unstable sets

We will try to find the analogue of the stable and unstable set for nonlinear autonomous systems $\dot{x} = f(x)$. First we define the **global stable set** of a fixed point x_0

$$W_s(x_0) = \{x \in \mathbb{R}^n : \lim_{t \to \infty} |\phi_t(x) - x_0| = 0\}$$

and $W_u(x_0)$, the **global unstable set**, is defined with time reversed. These are invariant sets. We would like to investigate their structure.

Basically what we will see, when the linearization at x_0 is hyperbolic, is that $W_s(x_0)$ and $W_u(x_0)$ are regular graphs over the stable and unstable subspace (respectively) in a neighborhood of x_0 and are tangential to $E_s(Df(x_0))$ and $E_u(Df(x_0))$ (respectively) at x_0.

Stable manifold theorem

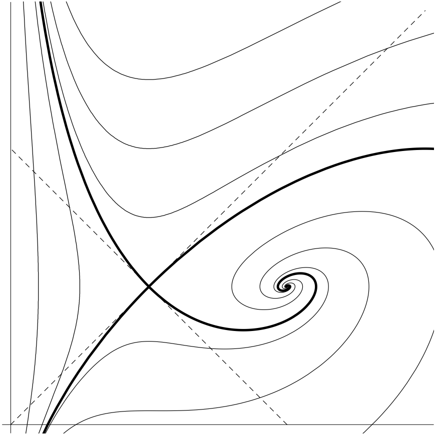


Figure: Global stable and unstable manifolds (in bold) of a hyperbolic fixed point for a planar nonlinear system. The linearized stable and unstable subspaces are drawn with dotted lines.

Figure from Teschl, p. 258

Local stable and unstable sets

We start out by looking at a smaller set with better convergence property. Fix a neighborhood U of x_0 and define

$$M_{s,\alpha}(x_0) = \{x : \Gamma_+(x) \subset U \ \text{ and } \ \sup_{t \geq 0} e^{\alpha t}|\phi_t(x) - x_0| < +\infty\}.$$

Then we define the **local stable manifold** at x_0 in the neighborhood U

$$M_s(x_0) = \cup_{\alpha>0} M_{s,\alpha}(x_0).$$

This is the set of initial data for which the corresponding solution stays in U for all $t > 0$ and converges exponentially to x_0 with some rate.

Stable manifold theorem

Theorem

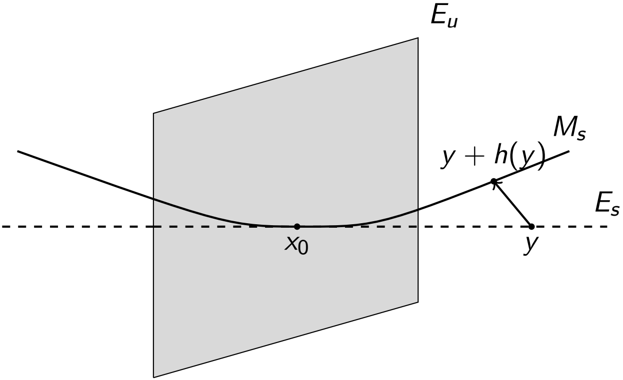
Suppose that $f \in C^k$ *has a fixed point at* 0 *with* $A = Df(0)$ *and* A *is hyperbolic. Then there is a neighborhood* U *of* 0 *and a* C^k *function* $h : E_s(A) \cap U \to E_u(A)$ *so that*

$$M_s(0) \cap U = \{y + h(y) : \; y \in E_s(A) \cap U\}.$$

Furthermore both $h(0) = 0$ *and* $Dh(0) = 0$*, i.e.* $M_s(0)$ *is tangent to the linearized stable subspace* $E_s(A)$ *at* 0*.*

The same holds for the unstable manifold by reversing time, with a function $h_u : U \cap E_u(A) \to E_s(A)$ instead. Let me also remark that, like the Picard theorem, the existence proof will be by contraction mapping and therefore will essentially give an algorithm to compute the stable/unstable manifold.

Graph over stable subspace

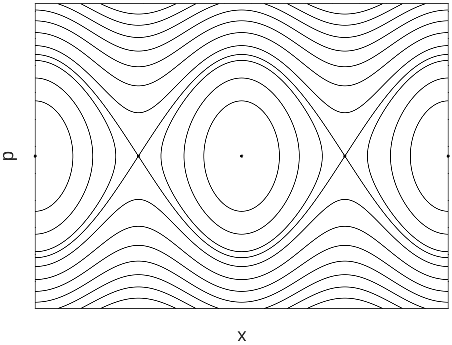


Pendulum system (explicit example)

Usually one cannot compute the stable/unstable manifold explicitly. But, at least for Hamiltonian systems in $2-d$ it is often possible.

For example consider the pendulum system

$$\begin{cases} \dot{x} = p \\ \dot{p} = -\sin(x) \end{cases} \quad \text{with Hamiltonian} \quad H(p,x) = \frac{1}{2}p^2 - \cos(x).$$



Pendulum system (explicit example)

System has fixed points at $p = 0$ and $x = k\pi$. Linearization is

$$Df(0, 2k\pi) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad Df(0, (2k+1)\pi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We will focus on the odd multiples of π where the fixed points are hyperbolic with eigenvalues $\lambda_\pm = \pm 1$. Eigenvectors are

$$v_\pm = \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}.$$

Usually this would be as far as one could go before applying stable/unstable manifold theorem, but because of the conserved quantity we can do more, we can look for the level set of H containing $(0, (2k+1)\pi)$ which is $\{H(p, x) = 1\}$

Pendulum system (explicit example)

Usually this would be as far as one could go before applying stable/unstable manifold theorem. Here because of the conserved quantity we can do more, we can look for the level set of H containing $(0, (2k+1)\pi)$ which is $\{H(p,x) = 1\}$.

Then we write that equation

$$1 = \frac{1}{2}p^2 - \cos(x)$$

and solve for p

$$p_\pm(x) = \pm\sqrt{2}\sqrt{1+\cos(x)}.$$

Pendulum system (explicit example)

Define $b_s, \; b_u : [0, 2\pi] \to \mathbb{R}$

$$b_s(x) = \begin{cases} p_+(x) & x \in [0,\pi] \\ p_-(x) & x \in [\pi, 2\pi] \end{cases} \quad \text{and} \quad b_u(x) = -b_s(x)$$

Then we can identify the global stable/unstable manifold for $(0, \pi)$

$$W_s((0,\pi)) = \{(x, b_s(x)) : x \in (0, 2\pi)\}$$

and

$$W_s((0,\pi)) = \{(x, b_u(x)) : x \in (0, 2\pi)\}.$$

Pendulum system (explicit example)

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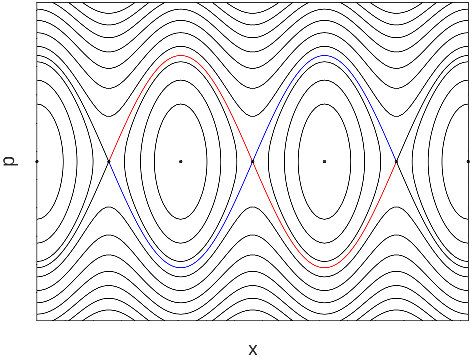


Pendulum with dissipation (non-explicit form)

After we prove the theorem I will give some more detailed applications, but let's point out now at least one application for the pendulum system with friction

$$\begin{cases} \dot{x} = p \\ \dot{p} = -\mu p - \sin(x). \end{cases}$$

Linearization at $(0, \pi)$ is

$$Df(0,\pi) = \begin{bmatrix} 0 & 1 \\ 1 & -\mu \end{bmatrix}$$

which has eigenvalues

$$\lambda_\pm = -\frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 + 4}.$$

The fixed point is always hyperbolic with one positive and one negative eigenvalue.

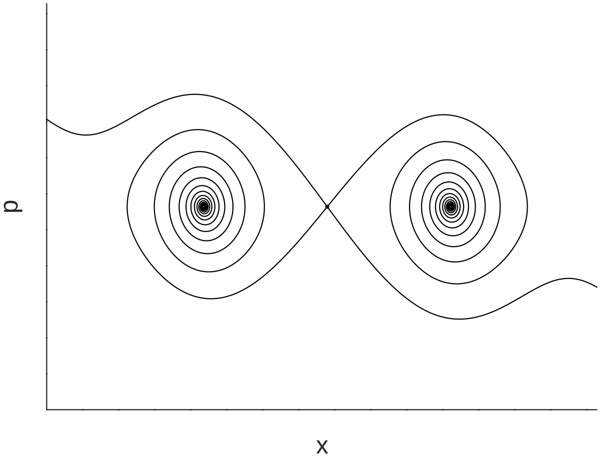
Pendulum with dissipation (non-explicit form)

Stable/unstable manifold theorem guarantees the existence of two solutions with $\omega_+(x) = (0, \pi)$ and two solutions with $\omega_-(x) = (0, \pi)$ which are, respectively, tangential to the eigenvectors of the linearization

$$v_\pm = \begin{bmatrix} 1 \\ 1/(\frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2+4}) \end{bmatrix}.$$

as they approach/leave the fixed point.

Pendulum with dissipation (non-explicit form)



Finding the stable manifold

We will think of the perturbed system of the form

$$\dot{x} = Ax + g(x)$$

where A is hyperbolic, and $g(0) = 0$ and $Dg(0) = 0$. Of course a general nonlinear autonomous system can be written in this form in a neighborhood of a hyperbolic fixed point.

As usual when we are taking a perturbative approach we want to use Duhamel

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-r)}g(x(r))\, dr.$$

Projectors

Call P_s and P_u to be the linear projectors onto the stable and unstable subspaces, and

$$x_u = P_u x,\ x_s = P_s x,\ g_u(x) = P_u g(x), \ \text{ and } \ g_s(x) = P_s g(x).$$

We are looking for initial data x_0 so that the solution $x(t)$ stays bounded for all positive times. If g was zero we would just take $P_u x_0 = x_u = 0$.

Instead given x_s we will try to find $x_u = h(x_s)$ so that the solution with initial data $x_0 = x_s + h(x_s)$ stays bounded.

Equation for $x_u(0)$

We project the Duhamel formula onto the unstable subspace, and note that eigenspace projectors for A commute with A,

$$x_u(t) = e^{At} x_u(0) + \int_0^t e^{A(t-r)} g_u(x(r)) \, dr.$$

Then we rearrange this solving for $x_u(0)$ which is our interest

$$x_u(0) = e^{-At} x_u(t) - \int_0^t e^{-Ar} g_u(x(r)) \, dr.$$

Eigenspace projector commutation

The eigenspace projectors can be expressed in terms of the resolvent operator $R : \mathbb{C} \to L(\mathbb{C}^n, \mathbb{C}^n)$ for the exponential e^A

$$R(\lambda) = (e^A - \lambda I)^{-1}.$$

This meromorphic function has a pole at each eigenvalue with residue being the projection onto the corresponding generalized eigenspace (can see this by Jordan normal form). Then the stable and unstable subspace projection operators for a hyperbolic matrix can be expressed by

$$P_s = \frac{1}{2\pi i} \int_{|\lambda|=1} R(\lambda)\, d\lambda \quad \text{and} \quad P_u = I - P_s.$$

Since this representation gives a convergent power series expansion in terms of e^A the operators both commute with A.

Equation for $x_u(0)$

Now we will try sending $t \to \infty$ in

$$x_u(0) = e^{-At} x_u(t) - \int_0^t e^{-Ar} g_u(x(r))\, dr.$$

Now since we are looking for an initial data so that $x(t)$ stays bounded for positive times, and $\|e^{-At} P_u\|_{op} \leq e^{-\alpha_0 t}$ for some $\alpha_0 > 0$,

$$x_u(0) = -\int_0^\infty e^{-Ar} g_u(x(r))\, dr$$

where the convergence of the integral follows, again, from the exponential decay of the integrand.

Integral equation on the stable manifold

Now we use this formula to write down an integral equation for solutions on the stable manifold

$$
\begin{aligned}
x(t) &= e^{At}x_0 + \int_0^t e^{A(t-r)}g(x(r))\,dr \\
&= e^{At}x_s(0) + \int_0^t e^{A(t-r)}g(x(r))\,dr - \int_0^\infty e^{A(t-r)}g_u(x(r))\,dr \\
&= e^{At}x_s(0) + \int_0^t e^{A(t-r)}g_s(x(r))\,dr - \int_t^\infty e^{A(t-r)}g_u(x(r))\,dr \\
&= e^{At}x_s(0) + \int_0^\infty e^{A(t-r)}P(t-r)g(x(r))\,dr
\end{aligned}
$$

where

$$P(t) = P_s 1_{t>0} + P_u 1_{t<0}.$$

Integral equation on the stable manifold

So to summarize we have found the integral equation

$$x(t) = e^{At}x_s(0) + \int_0^\infty e^{A(t-r)}P(t-r)g(x(r))\ dr$$

where the operator in the integrand satisfies

$$\|e^{At}P(t)\|_{op} \leq Ce^{-\alpha_0|t|}$$

for some

$$0 < \alpha_0 < \min\{|\mathrm{Re}(\lambda)| : \ \lambda \in \sigma(A)\}.$$

We will find solutions of this integral equation via contraction mapping! We will need smallness somewhere to prove the contraction property, it cannot come from small time interval anymore, instead it will come from small initial data.

Contraction mapping set up

The underlying Banach space, similar to the Picard theorem but now on an infinite time interval, is

$$C([0,\infty) \to \mathbb{R}^n) = \{\gamma : [0,\infty) \to \mathbb{R}^n : \ \gamma \ \text{ continuous}\}$$

with the supremum norm

$$\|\gamma\|_{sup} = \sup_{t \geq 0} |\gamma(t)|.$$

We will work in the subset

$$X = \{\gamma \in C([0,\infty) \to \mathbb{R}^n) : \ P_s\gamma(0) = P_s x_0 \ \text{ and } \ \|\gamma\|_{sup} \leq \delta\}.$$

Note that this subset only specifies the stable component of the initial data, the unstable component needs to be solved for by the fixed point argument. We will see which particular $\delta > 0$ we need in the course of the proof.

Contraction mapping set up

The mapping in question will be chosen

$$\Phi[\gamma](t) := e^{At}x_s(0) + \int_0^\infty e^{A(t-r)}P(t-r)g(\gamma(r))\, dr.$$

We need to check that Φ is well defined and bounded on $[0,\infty)$, $\Phi[\gamma]$ is continuous, and $\Phi : X \to X$ i.e.

$$P_s\Phi[\gamma](0) = P_s x_0 \ \text{ and } \ \|\Phi[\gamma]\|_{sup} \leq \delta$$

when $\gamma \in X$.

Contraction mapping set up

Choice of $\delta > 0$

Since $Dg(0) = 0$ and g is C^1 for all $\varepsilon > 0$ there is a $\delta > 0$ so that so that

$$|g(x) - g(y)| \leq \varepsilon |x - y| \ \text{ for all } \ |x|, |y| \leq \delta$$

also, in particular with $y_0 = 0$ using $g(0) = 0$,

$$|g(x)| \leq \varepsilon |x| \ \text{ for } \ |x| \leq \delta.$$

We will use $\varepsilon = \frac{\alpha_0}{4C}$ which then fixes $\delta > 0$ as well.

We will also specify now that $|x_s(0)| \leq \delta/2C$ so that

$$|e^{At} x_s(0)| \leq Ce^{-\alpha_0 t} |x_s(0)| \leq \frac{\delta}{2}.$$

Contraction mapping set up

Map well-defined and $\Phi[\gamma]$ is bounded

Now we look for an upper bound on the integral term in the definition of $\Phi[\gamma]$

$$
\begin{aligned}
|e^{A(t-r)}P(t-r)g(\gamma(r))| &\leq \|e^{A(t-r)}P(t-r)\|_{op}|g(\gamma(r))| \\
&\leq Ce^{-\alpha_0|t-r|}\varepsilon\|\gamma\|_{sup}
\end{aligned}
$$

so

$$
\begin{aligned}
\left|\int_0^\infty e^{A(t-r)}P(t-r)g(\gamma(r))\,dr\right| &\leq C\varepsilon\|\gamma\|_{sup}\int_{-\infty}^\infty e^{-\alpha_0|t|}dt \\
&= \frac{2C}{\alpha_0}\varepsilon\|\gamma\|_{sup}
\end{aligned}
$$

Contraction mapping set up

Map well-defined and $\Phi[\gamma]$ is bounded

Combining those previous estimates

$$|\Phi[\gamma](t)| \leq C|x_0| + \frac{2C}{\alpha_0}\varepsilon\delta \leq \frac{\delta}{2} + \frac{\delta}{2} \leq \delta$$

so $\Phi[\gamma]$ is well defined, bounded and even

$$\|\Phi[\gamma]\|_{sup} \leq \delta.$$

Contraction mapping set up

$\Phi[\gamma](t)$ is continuous in t

The term $e^{At}x_s(0)$ is continuous in t. We just need to be slightly careful with the integral term because $P(t)$ is not continuous, the continuity is more clear if we expand back out to

$$\int_0^t e^{A(t-r)}P_s g(\gamma(r))\, dr + \int_t^\infty e^{A(t-r)}P_u g(\gamma(r))\, dr.$$

Both terms above are continuous because they are the product of a continuous function e^{At} with the anti-derivative of a continuous function.

Contraction mapping set up

Initial data property

Finally we must check the initial data property, we plug in $t = 0$ and find

$$P_s\Phi[\gamma](0) = x_s(0) + \int_0^\infty e^{-Ar} P_s P(-r) g(\gamma(r))\, dr$$

note that $P(-r) = P_u$ for all $r > 0$ and $P_s P_u = 0$ so

$$P_s\Phi[\gamma](0) = x_s(0) = P_s x_0.$$

Contraction property

Finally we aim to show the contraction property. Recall

$$\Phi[\gamma](t) := e^{At}x_s(0) + \int_0^\infty e^{A(t-r)}P(t-r)g(\gamma(r))\, dr.$$

So we compute, using the choice $\varepsilon = \frac{\alpha_0}{4C}$,

$$\begin{aligned}
|\Phi[\gamma](t) - \Phi[\rho](t)| &= |\int_0^\infty e^{A(t-r)}P(t-r)(g(\gamma(r)) - g(\rho(r))\, dr| \\
&\leq \int_0^\infty \|e^{A(t-r)}P(t-r)\|_{op}|g(\gamma(r)) - g(\rho(r))|\, dr \\
&\leq \int_0^\infty Ce^{-\alpha_0|t-r|}\varepsilon|\gamma(r) - \rho(r)|\, dr \\
&\leq \frac{2C}{\alpha_0}\varepsilon\|\gamma - \rho\|_{sup} \\
&\leq \frac{1}{2}\|\gamma - \rho\|_{sup}.
\end{aligned}$$

Analyzing the fixed point

So we have established that there is a neighborhood U of 0 so that for every $v \in E_s(A) \cap U$ there is a *unique* solution $x(t, v) \in X$ of

$$x(t, v) = e^{At}v + \int_0^\infty e^{A(t-r)}P(t-r)g(x(r, v))\, dr$$

which is, in particular, a solution also of

$$\dot{x} = Ax + g(x)$$

with

$$x(0, v) = v + \int_0^\infty e^{-Ar}P_u g(x(r, v))\, dr$$

and we can define

$$h(v) = \int_0^\infty e^{-Ar}P_u g(x(r, v))\, dr.$$

Analyzing the fixed point

We can read off of the equation

$$x(t,v) = e^{At}v + \int_0^\infty e^{A(t-r)}P(t-r)g(x(r,v))\,dr$$

that $x(t,0) = 0$ solves with $v = 0$ (since $g(0) = 0$) so

$$h(0) = 0.$$

Let's assume that $h(v)$ is differentiable and compute $D_v h(0)$

$$D_v h(v) = \int_0^\infty e^{-Ar}P_u Dg(x(r,v))D_v x(r,v)\,dr$$

but $Dg(x(r,0)) = Dg(0) = 0$ so

$$D_v h(0) = 0.$$

Differentiability of h

To justify the differentiability, and higher order differentiability, of

$$h(v) = \int_0^\infty e^{-Ar} P_u g(x(r,v))\, dr.$$

the main point is to show that $x(r,v)$ is differentiable in v.

This is a general fact one can prove about integral equations of the form

$$x(t,v) = e^{At}v + \int_0^\infty e^{A(t-r)} P(t-r) g(x(r,v))\, dr,$$

basically $x(t,v)$ will be differentiable in v as many times as g is differentiable. We skip the details at least for now.

Summarizing what we have proved

Notice that, in the course of the proof, we showed that if $x(t)$ is *any* solution so that $\Gamma_+(x(0)) \subset U$ then x must solve

$$x(t) = e^{At}x_s(0) + \int_0^\infty e^{A(t-r)}P(t-r)g(x(r))\,dr$$

and as a consequence, as long as the neighborhood U is sufficiently small so that this fixed point problem has a unique solution, then

$$x_u(0) = h(x_s(0)).$$

So

$$W_s(0) \cap U = M_s(0, U) = \{y + h(y) : y \in E_s(A) \cap U\}.$$

Non-hyperbolic systems

We can still identify the local (exponentially) stable and unstable manifolds without hyperbolicity, although the identification with the global stable and unstable sets will not hold anymore.

The idea is to make a change of variables to

$$y(t) = e^{\beta t} x(t) \text{ for some } \beta > 0 \text{ small.}$$

Then y solves

$$\begin{aligned} \dot{y} &= \beta e^{\beta t} x + e^{\beta t}(Ax + g(x)) \\ &= (A + \beta I) y + e^{\beta t} g(e^{-\beta t} y). \end{aligned}$$

When $\beta > 0$ is sufficiently small there will be no eigenvalues of $(A + \beta I)$ on the imaginary axis.

Non-hyperbolic systems

For this "tilted" equation

$$\dot{y} = (A + \beta I)y + e^{\beta t} g(e^{-\beta t} y)$$

we define the analogous integral equation

$$y(t) = e^{(A+\beta)t} y_s(0) + \int_0^\infty e^{(A+\beta)(t-r)} P(t-r) e^{\beta t} g(e^{-\beta t} y(r))\, dr$$

and, assuming there is a fixed point for $y_s \in E_s(A) \cap U$, we define a map

$$h_\beta : E_s(A) \cap U \to (E_c(A) \oplus E_u(A)) \cap U$$

by solving the above integral equation with initial data y_s and writing

$$h_\beta(y_s) = P_u y(0) + P_c y(0).$$

Integral equation for the tilted equation

The integral equation

$$y(t) = e^{(A+\beta)t} y_s(0) + \int_0^\infty e^{(A+\beta)(t-r)} P(t-r) e^{\beta t} g(e^{-\beta t} y(r))\, dr$$

still satisfies the assumptions needed to apply the fixed point argument

$$|e^{\beta t} g(e^{-\beta t} y) - e^{\beta t} g(e^{-\beta t} y')| \leq \varepsilon |y - y'|$$

for $y \in U$ and $e^{\beta t} g(e^{-\beta t} 0) = 0$.

Also note that if y is a bounded solution of the above then $|x(t)| = e^{-\beta t}|y(t)|$ converges to the fixed point at 0 with exponential rate β, i.e. $x(0) = y(0) \in M_{s,\beta}(0)$.

Non-hyperbolic systems

Thus we have found a C^k map

$$h_\beta : E_s(A) \cap U \to (E_c(A) \oplus E_u(A)) \cap U$$

with $h_\beta(0) = 0$ and $Dh_\beta(0) = 0$ and

$$M_{s,\beta}(0) \cap U = \{x : \Gamma_+(x) \subset U \ \text{ and } \ \sup_{t \geq 0} e^{\beta t} |\phi_t(x)| < +\infty\}$$
$$= \{y + h_\beta(y) : y \in E_s(A) \cap U\}.$$

This definition will be independent of β for $\beta > 0$ smaller than $\alpha_0 = \min |\mathrm{Re}(\lambda_j)|$ and so we have also found a representation of the local stable manifold $M_s(0)$.

Parameter dependence

If the equation depends on a parameter $\mu \in M$ (some parameter space in $\mathbb{R}^m$)

$$\dot{x} = A(\mu)(x - x_0(\mu)) + g(x, \mu)$$

and $A(0)$ is hyperbolic then $A(\mu)$ is hyperbolic in a neighborhood of $\mu = 0$ and:

Theorem

There is a neighborhood U of $x_0(0)$ and a function

$$h_s : E_s \cap U \times M \to E_u$$

which is C^k in all variables such that

$$M_s(x_0(\mu)) \cap U = \{x_0(\mu) + P_s(\mu)y + P_u(\mu)h_s(y, \mu) : \; y \in E_s \cap U\}.$$

Application of stable / unstable manifold theorem to phase transitions / travelling waves

Finding heteroclinics / homoclinics

A solution leaving a hyperbolic fixed point along the unstable manifold may end up in the stable manifold approaching another fixed point as $t \to \infty$, this would be a heteroclinic/homoclinic orbit.

In some cases we can take advantage of the stable/unstable manifold theorem as part of a proof of existence of heteroclinic/homoclinic orbits especially in 2-d.

Phase transitions for a Allen-Cahn equation

Consider the following PDE which is called the **Allen-Cahn equation** or a **bistable reaction diffusion equation**

$$\partial_t u - \partial_{xx}^2 u = f(u).$$

The reaction term is **bistable**

$$f(u) = -W'(u)$$

where W is a symmetric double well potential

Bistable nonlinearity

Figure: Plot of symmetric double well potential $W(u) = \frac{1}{4}(1 - x^2)^2$.

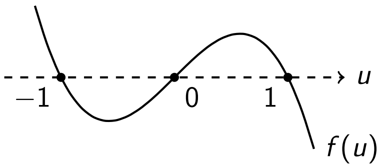


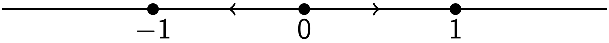
Figure: Plot of bistable nonlinearity $f(u) = -W'(u)$.

Phase line analysis of x-independent solutions

If $u(x,t) = u(t)$ is an x-independent solution of the reaction diffusion equation then it simply solves the 1-d ODE

$$\dot{u} = f(u)$$

with the phase line



We just care about solutions in $[-1,1]$ which is the physical range for this model, there is an unstable critical point at 0 and two stable critical points at $(-1,0)$ and $(1,0)$.

Phase transition solutions

Of course x-independent solutions are not very interesting from the PDE perspective. We look for a **transition solution** which links -1 to 1

$$u(x,t) = \xi(x)$$

with

$$\lim_{s \to -\infty} \xi(s) = -1, \ \lim_{s \to \infty} \xi(s) = 1, \ \text{ and } \ \lim_{s \to \pm\infty} \dot{\xi}(s) = 0.$$

Let's find what ODE is solved by ξ.

Phase transition ODE

If we plug in the form $\xi(x)$ into the reaction diffusion equation we find

$$-\ddot{\xi} = f(\xi)$$

which is familiar type of ODE. Let's write it as a first order system

$$\frac{d}{dt}\left[\begin{array}{c} \xi \\ \dot{\xi} \end{array}\right] = \left[\begin{array}{c} \dot{\xi} \\ -f(\xi) \end{array}\right].$$

This is a Hamiltonian system for

$$H(\dot{\xi}, \xi) = \frac{1}{2}\dot{\xi}^2 - W(\xi).$$

Contour plot of the Hamiltonian

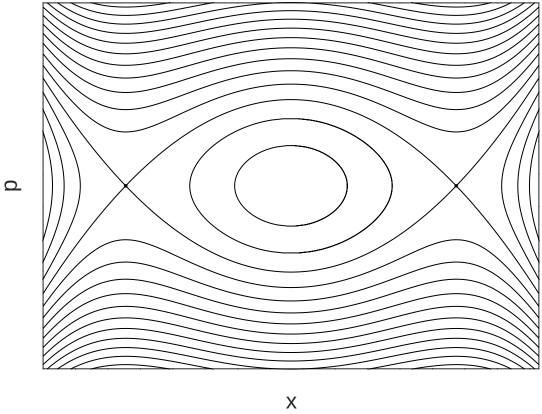


Figure: Contour plot for $H(\dot{\xi}, \xi) = \frac{1}{2}\dot{\xi}^2 - W(\xi)$.

Heteroclinics

The fixed points $(-1,0)$ and $(1,0)$ lie on the same level set of H, $H(-1,0) = H(1,0) = 0$

$$\frac{1}{2}\dot{\xi}^2 = \frac{1}{4}(1-\xi^2)^2$$

so

$$\dot{\xi} = \pm\frac{1}{\sqrt{2}}(1-\xi^2).$$

Since we are looking for the heteroclinic from -1 to $+1$ we choose the positive branch. This first order ODE can be solved by separation of variables and some calculus

$$\xi(s) = \tanh\left(\frac{s}{\sqrt{2}}\right).$$

Asymmetric bistable potentials

The situation, at least vis-a-vis the ODE heteroclinic problem, becomes more interesting when the potential is asymmetric.

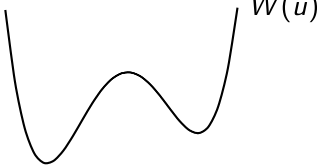


Figure: Plot of double well potential $W(u)$.

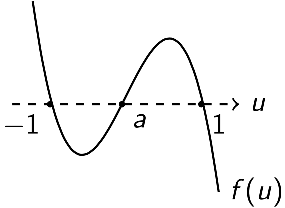


Figure: Plot of asymmetric bistable nonlinearity $f(u) = -W'(u)$.

Phase transition solutions

When the potential is asymmetric both -1 and 1 are stable but -1 has lower energy so we expect that there is a **travelling wave solution** moving to the right with positive speed linking -1 to 1 (the negative phase invades the positive phase)

$$u(x,t) = \xi(x - ct)$$

with

$$\lim_{s\to-\infty} \xi(s) = -1, \ \lim_{s\to\infty} \xi(s) = 1, \ \text{ and } \ \lim_{s\to\pm\infty} \dot{\xi}(s) = 0$$

for some $c > 0$. We will need to identify both the solution ξ and the speed $c > 0$.

Let's find what ODE is solved by ξ.

Travelling wave picture

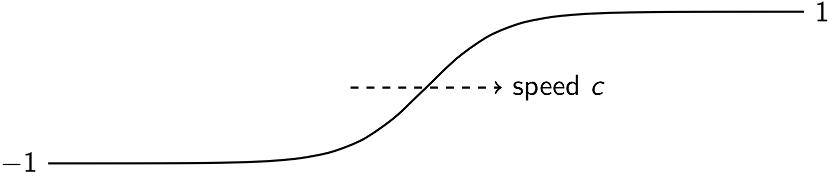


Figure: graph of $u(x, t)$ at some fixed time.

Travelling wave ODE

If we plug in the form $\xi(x - ct)$ into the reaction diffusion equation we find

$$-c\dot{\xi} - \ddot{\xi} = f(\xi)$$

or as a first order system

$$\frac{d}{dt}\begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \dot{\xi} \\ -c\dot{\xi} - f(\xi) \end{bmatrix}.$$

This is a similar Hamiltonian system to before but now with a dissipative term, again the energy is

$$H(\dot{\xi}, \xi) = \frac{1}{2}\dot{\xi}^2 - W(\xi).$$

Note that this Hamiltonian is a bit different from previous ones we have studied – the potential $-W(\xi) \to -\infty$ as $|\xi| \to \infty$ so dissipation may drive solutions off to ∞.

Asymmetric phase plane when $c = 0$

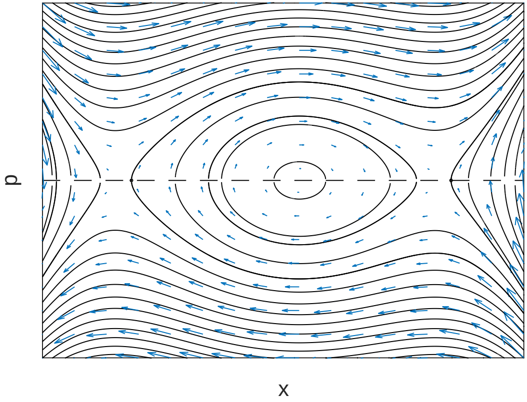


Figure: Contour plot for $H(\dot{\xi}, \xi) = \frac{1}{2}\dot{\xi}^2 - W(\xi)$. Dots mark the fixed points, the dashed line is the $\dot{\xi} = 0$ axis which the system is symmetric with respect to.

Linearizations

The system

$$\frac{d}{dt}\begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \dot{\xi} \\ -c\dot{\xi} - f(\xi) \end{bmatrix} = F_c(\xi, \dot{\xi}).$$

has fixed points at $(-1, 0)$, $(a, 0)$ and $(0, 1)$.

The fixed point at $(a, 0)$ is at a local minimum of H and is asymptotically stable. We are interested in the fixed points at $(-1, 0)$ and $(1, 0)$ which are saddles with linearizations

$$DF_c(\pm 1, 0) = \begin{bmatrix} 0 & 1 \\ -f'(\pm 1) & -c \end{bmatrix}.$$

Orbits of interest

When $c > 0$ the system is not integrable but the stable/unstable manifold theorem still guarantees the existence of an orbit $\Gamma_{-1}(c) \subset W_u((-1,0))$ leaving $(-1,0)$ tangential to the unstable direction

$$v_u(-1) = \begin{bmatrix} 1 \\ \frac{1}{2}(-c + \sqrt{c^2 - 4f'(-1)}) \end{bmatrix}$$

and an orbit $\Gamma_1(c) \subset W_s((1,0))$ approaching $(1,0)$ tangential to the stable direction

$$v_s(1) = \begin{bmatrix} -1 \\ \frac{1}{2}(c + \sqrt{c^2 - 4f'(1)}) \end{bmatrix}$$

Furthermore these orbits vary continuously with respect to the parameter c. The goal is to find a value of c so that these two curves are the same.

Note that the second component of $v_u(-1)$ is decreasing as c increases and the second component of $v_s(1)$ is increasing as c increases.

Orbits of interest

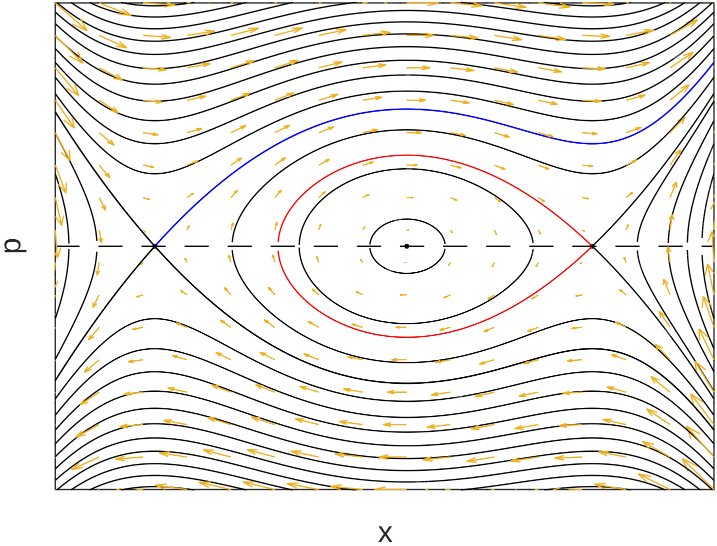


Figure: $\Gamma_1(c)$ red and $\Gamma_{-1}(c)$ in blue when $c = 0$.

Middle line

Call L to be the vertical line through $(a, 0)$. We claim that both $\Gamma_1(c)$ and $\Gamma_{-1}(c)$ intersect $L \cap \{p > 0\}$ at a unique point. Taking that for granted for the moment we define

$$\gamma_{\pm 1}(c) = \Gamma_{\pm 1}(c) \cap L \cap \{p > 0\}.$$

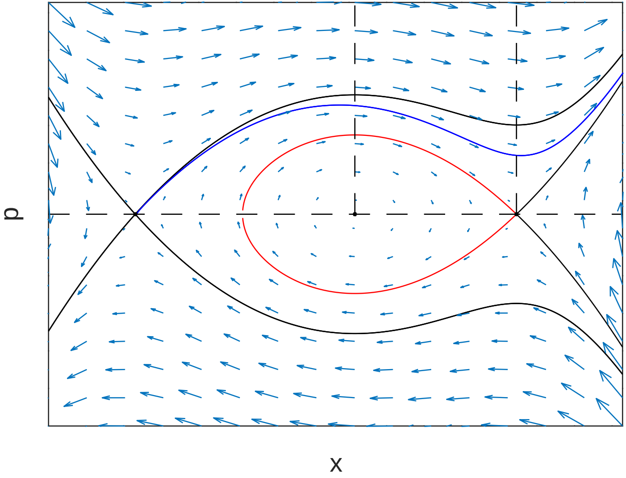
By the energy inequality

$$\frac{d}{dt} H(\dot{\xi}, \xi) = -c|\dot{\xi}|^2 \leq 0$$

the curve $\Gamma_{-1}(c)$ has to stay inside the region bounded by $\Gamma_{-1}(0)$ and it's reflection accross the x-axis so

$$\gamma_{-1}(c) \leq \gamma_{-1}(0).$$

Orbits of interest when $c > 0$ small



Pushing up $\gamma_1(c)$

Consider the line segment

$$R = \{(p, x) : \; x \in [a + \varepsilon, 1] \;\text{ and }\; p = \beta(1 - x)\}.$$

We will show that $\Gamma_1(c)$ cannot cross R before it crosses L (backwards in time), this amounts to checking that the vector field on R points in the correct direction

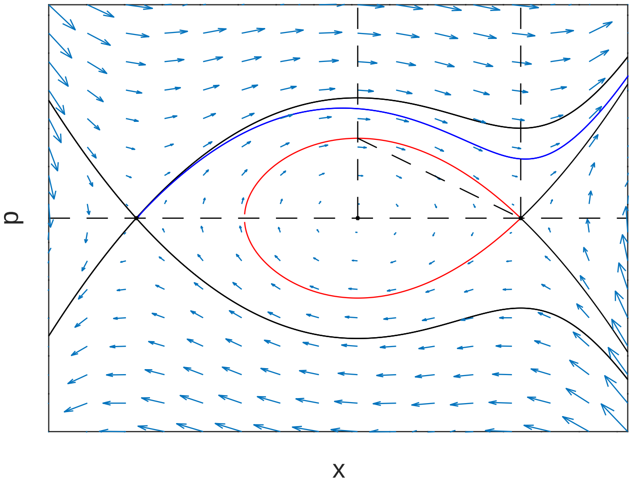
$$\dot{x} = \beta(1 - x) \;\text{ and } \dot{p} = -c\beta(1 - x) - f(x)$$

so with $n = (\beta, 1)$ an upward normal to R

$$-[\beta, 1]\cdot[\beta(1-x), -c\beta(1-x)-f'(x)] = -\beta^2(1-x)+c\beta(1-x)+f(x)$$

since $f(x)/(1 - x)$ is bounded on $(a, 1)$ this can be made positive by choosing $c > 0$ large.

Pushing up $\gamma_1(c)$



Flow conjugacy and the Hartman-Grobman theorem

Conjugacy

We have already seen several ways in which flows of different vector fields "look the same" as a linear flow. We want to make this idea more precise now using the idea of **conjugacy**.

Let's start out with something simple, the linear case, we say that a pair of flow maps ϕ_t and ψ_t are **linearly conjugate** if there is an invertible matrix Q so that

$$Q\psi_t(x) = \phi_t(Qx).$$

We already know a pair of linear flow maps e^{At} and e^{Bt} are linearly conjugate if $A = Q^{-1}BQ$ for some invertible Q. This is an if and only if because

$$Ax = \frac{d}{dt}\bigg|_{t=0} e^{At}x = \frac{d}{dt}\bigg|_{t=0} Q^{-1}e^{Bt}Qx = Q^{-1}BQx.$$

More general notions of conjugacy

In greater generality we say that a pair of flow maps ϕ_t and ψ_t are (linearly, C^k/differentiably, topologically) **conjugate** in respective neighborhoods U and V if there is a (linearly, C^k-ly, continuous-ly) invertible map $\mathfrak{h} : V \to U$ so that

$$\mathfrak{h} \circ \psi_t = \phi_t \circ \mathfrak{h}.$$

The notion of flow map conjugacy in a neighborhood of a point x_0 is an equivalence relation on the set of autonomous vector fields.

Conjugacy near non-stationary points

It turns out that all smooth flows which do not have a stationary point at some x_0 are differentiably conjugate in a neighborhood of x_0. In this sense the only "interesting" local behavior is near fixed points.

Lemma

Suppose that f is C^k and x_0 is a non-stationary point of the flow then there is a coordinate transform $\mathfrak{g} : U \to V$ (neighborhoods of x_0) so that the flow $y(t) = \mathfrak{g}(x(t))$ is

$$\dot{y} = [1, 0, \dots, 0].$$

That is, with ϕ_t the flow map for f and ψ_t the flow map for $[1, 0, \dots, 0]^T$ we have

$$\mathfrak{g}(\phi_t(x_0)) = \psi_t(\mathfrak{g}(x_0)) \text{ for } x_0 \in U, \ \phi_t(x_0) \in V.$$

Straightening out the flow

Call $e_1 = [1, 0, \dots, 0]$, we can assume $0 \in U$ and $f(0) = e_1$. The mapping $\mathfrak{g}$ we are looking for should send

$$\phi_t([0, x_2, \dots, x_n]) \mapsto [0, x_2, \dots, x_n] + t[1, 0, \dots, 0] = [t, x_2, \dots, x_n].$$

In other words it should be the inverse of the map

$$\mathfrak{h}([x_1, \dots, x_n]) = \phi_{x_1}([0, x_2, \dots, x_n]).$$

We need to check that this mapping is actually invertible. Note that

$$D\mathfrak{h}(0) = \left[\dot{\phi}_t, \frac{\partial \phi_t}{\partial x_2}, \dots, \frac{\partial \phi_t}{\partial x_2}\right]_{(t,x)=(0,0)}$$

Since $\phi_0(x) = x$ we find $\left.\frac{\partial \phi_t}{\partial x_j}\right|_{(t,x)=(0,0)} = e_j$ and $\dot{\phi}_0(0) = f(0) = e_1$ so $D\mathfrak{h}(0) = I_{n \times n}$.

Straightening out the flow

By inverse function theorem $\mathfrak{h}$ is an invertible mapping $V \to U$ neighborhoods of 0 and $\mathfrak{h}(0) = 0$ respectively. The inverse mapping $\mathfrak{g} = \mathfrak{h}^{-1}$ is also C^k.

Then, calling $y(t) = \mathfrak{g}(x(t))$, note that $D\mathfrak{h}(y)e_1 = \frac{\partial \mathfrak{h}}{\partial y_1} = f(\mathfrak{h}(y))$ so

$$\dot{y}(t) = \frac{d}{dt}\mathfrak{g}(x(t)) = D\mathfrak{g}(x(t))\dot{x}(t) = (D\mathfrak{h}(y(t)))^{-1} f(x(t)) = e_1.$$

□

Differentiable conjugacy near stationary points

Near stationary points differentiable conjugacy is too strong. For example consider the two flows

$$\dot{x} = x \quad \text{and} \quad \dot{y} = 2y$$

and corresponding flow maps

$$\phi_t(x) = e^{-t}x \quad \text{and} \quad \psi_t(x) = e^{-2t}x.$$

There is a natural conjugacy via the mapping $\mathfrak{g}(x) = \mathrm{sgn}(x)|x|^{1/2}$

$$(\mathfrak{g} \circ \psi_t)(x) = e^{-t}\mathrm{sgn}(x)|x|^{1/2} = (\phi_t \circ \mathfrak{g})(x)$$

but this map is a homeomorphism not a diffeomorphism.

Differentiable conjugacy near stationary points

In general we can say

Lemma

If two C^k flows $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are differentiably conjugate near respective fixed points x_0 and y_0 then $Df(x_0)$ and $Dg(y_0)$ are linearly conjugate.

Topological conjugacy is less strict

Lemma

Two hyperbolic linear flows $\dot{x} = Ax$ and $\dot{y} = By$ are topologically conjugate if and only if the dimension of the respective stable subspaces agree (or, equivalently, the dimension of the respective unstable subspaces agree).

Topological conjugacy

Thus we have settled that if we want a flexible notion of conjugacy near stationary points we should try **topological conjugacy**, the following theorem confirms that this idea does work at the nonlinear level as well:

Theorem (Hartman-Grobman)

Suppose f is a C^1 vector field with 0 as a hyperbolic fixed point. Call $\phi_t(x)$ to be the corresponding flow and $A = Df(0)$ the hyperbolic linearization at 0. Then there is a homeomorphism $\mathfrak{h}$ so that

$$\mathfrak{h} \circ e^{At} = \phi_t \circ \mathfrak{h}$$

in a sufficiently small neighborhood of 0.

No differentiable conjugacy

Also proved by Hartman, there is no C^1 linearizing conjugacy at the origin for the system

$$\dot{x} = ax, \ \dot{y} = (a-b)y + cxz, \ \dot{z} = -bz$$

with $a > b > 0$ and $c \neq 0$.

There are some results in the other direction, linearly stable/unstable stationary points of C^2 vector fields have a C^1 conjugacy with the linearization, as do C^2 planar vector fields at hyperbolic stationary points.

Topological conjugacy

In combination with the topological conjugacy result for linear flows we obtain:

Corollary

Suppose f and g are both C^1 vector fields with 0 as a hyperbolic fixed point and the dimension of the respective stable manifolds agree then the corresponding flows are topologically conjugate in a neighborhood of 0.

Proof of differentiable conjugacy characterization

Let's start proving some of these results, first

Lemma

If two C^k flows $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are differentiably conjugate near respective fixed points x_0 and y_0 then $Df(x_0)$ and $Dg(y_0)$ are linearly conjugate.

Proof. Let $\mathfrak{h}$ be the diffeomorphism mapping a neighborhood V of y_0 onto a neighborhood V of x_0 with

$$\mathfrak{h} \circ \psi_t = \phi_t \circ \mathfrak{h}.$$

In particular note that $\mathfrak{h}(y_0) = x_0$. Then we compute the y derivative at y_0

$$D\mathfrak{h}(y_0) D\psi_t(y_0) = D\phi_t(x_0) D\mathfrak{h}(y_0).$$

Proof of differentiable conjugacy characterization

Next we compute the t derivative of

$$D\mathfrak{h}(y_0)D\psi_t(y_0) = D\phi_t(x_0)D\mathfrak{h}(y_0)$$

and use the linearized equation

$$\frac{d}{dt}D\psi_t(y_0) = Dg(y_0)D\psi_t(y_0) \text{ and } \frac{d}{dt}D\phi_t(x_0) = Df(x_0)D\psi_t(x_0)$$

to find

$$D\mathfrak{h}(y_0)e^{Dg(y_0)t} = e^{Df(x_0)t}D\mathfrak{h}(y_0).$$

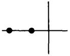
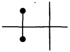
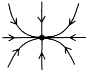
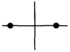
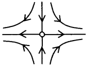
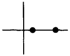
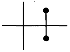
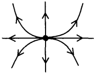
So the linearized flows are linearly conjugate. □

Topological equivalence of hyperbolic linear systems

Lemma

Two hyperbolic linear flows $\dot{x} = Ax$ and $\dot{y} = By$ are topologically conjugate if and only if the dimension of the respective stable subspaces agree (or, equivalently, the dimension of the respective unstable subspaces agree).

Topological classes of hyperbolic planar fixed points

(n_+, n_-)	Eigenvalues	Phase portrait		Stability
(0, 2)	 	 	node focus	stable
(1, 1)			saddle	unstable
(2, 0)	 	 	node focus	unstable

Kuznetsov, Elements of Applied Bifurcation Theory, Figure 2.5.

Conjugacy of linear systems: node focus equivalence

We already saw what can happen in $d = 1$, let's consider a specific multidimensional case of the two systems

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x \ \text{ and } \ \dot{y} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} y.$$

The first system has a stable node at the origin with a repeated (but non-degenerate) eigenvalue $\lambda = -1$ and the second has a stable focus with eigenvalues

$$\lambda_{\pm} = -1 \pm i.$$

Conjugacy of linear systems: node focus equivalence

The corresponding flow maps are, for the node,

$$\phi_t(x) = e^{-t}x$$

and, for the focus,

$$\psi_t(x) = R(t)e^{-t}x$$

where R is the rotation matrix

$$R(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

Conjugacy of linear systems: node focus equivalence

Consider a point x in polar coordinates (r_0, θ_0). For the node system the orbit containing this point passes through $(1, \theta_0)$ on the unit circle and takes time

$$\tau(r_0) = -\log(r_0)$$

to reach x.

Now consider the point y which is the forward flow of $(1, \theta_0)$ under the focus evolution

$$r(y) = r_0 \quad \text{and} \quad \theta(y) = \theta_0 + \tau(r_0) = \theta_0 - \log(r_0).$$

This will be our mapping defined in polar coordinates

$$h(r_0, \theta_0) = (r_0, \theta_0 - \log(r_0)).$$

Conjugacy of linear systems: node focus equivalence

Let's call $\mathfrak{h}$ to be the mapping h in Cartesian coordinates. Now, by construction,

$$\mathfrak{h} \circ \phi_t(x) = \psi_t \circ \mathfrak{h}(x)$$

for all $|x| = 1$ (note that $\mathfrak{h}(x) = x$ for $|x| = 1$).

However if we write $x = \phi_s(z)$ with $|z| = 1$ then $\mathfrak{h}(x) = \psi_s(z)$ by definition and

$$\mathfrak{h} \circ \phi_t(x) = \mathfrak{h} \circ \phi_{t+s}(z) = \psi_{t+s} \circ \mathfrak{h}(z) = \psi_t(\psi_s(z)) = \psi_t \circ \mathfrak{h}(x).$$

Conjugacy of linear systems: node focus equivalence

The mapping

$$h(r_0, \theta_0) = (r_0, \theta_0 - \log(r_0))$$

is continuous including as $r_0 \to 0$ and it is invertible because it acts on each circle as a rotation.

However the mapping $\mathfrak{h}$ in Cartesian coordinates is not differentiable at the origin

$$\mathfrak{h}(x_1, x_2) = R(-\frac{1}{2}\log(x_1^2 + x_2^2)) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The Poincaré map

$(n-1)$-dimensional surfaces

A set $\Sigma \subset \mathbb{R}^n$ is called an $(n-1)$-dimensional surface, or a **submanifold** of codimension one, if it can be written as

$$\Sigma = \{x \in U : S(x) = 0\}$$

where U is open, S is smooth, and $\nabla S(x) \neq 0$ for all $x \in \Sigma$. This is called a **level set representation** of the surface Σ.

In this case the **normal direction** to Σ at a point $x \in \Sigma$ is

$$n_x = \frac{\nabla S(x)}{|\nabla S(x)|}$$

(note that the choice of S vs $-S$ prescribes an orientation on Σ).

Transversal submanifolds

A surface Σ is said to be **transversal** to the vector field $f(x)$ if

$$f(x) \cdot n_x \neq 0 \quad \text{for all} \quad x \in \Sigma.$$

If a point x gets pushed through a transversal manifold Σ by the flow then points in a neighborhood will also pass through Σ at a nearby time.

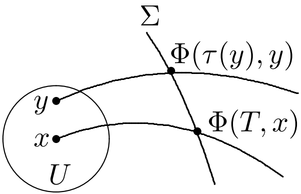
Theorem

Suppose $x \in \mathbb{R}^n$ and Σ transversal such that $\phi_T(x) \in \Sigma$. Then there is a neighborhood U of x and a smooth mapping $\tau : U \to \mathbb{R}$ such that $\tau(x) = T$ and for all $y \in U$,

$$\phi_{\tau(y)}(y) \in \Sigma.$$

Transversal submanifolds

Proof



We are trying to solve for t in the equation

$$S(\phi_t(y)) = 0$$

in a way which is continuous in y. We have a single solution at (T, x). This is a natural place to use the implicit function theorem.

Figure from Teschl, page 197

Transversal submanifolds

Proof

Note that

$$\frac{\partial}{\partial t}S(\phi_t(y)) = \nabla S(\phi_t(y)) \cdot \frac{\partial}{\partial t}\phi_t(y) = \nabla S(\phi_t(y)) \cdot f(\phi_t(y)) \neq 0$$

for (t, y) in a neighborhood of (T, x) by transversality and continuity of f and ∇S.

Thus implicit function theorem gives a map $\tau(y)$, which is as smooth as $S(\phi_t(y))$, defined on a neighborhood U of x such that

$$S(\phi_{\tau(y)}(y)) = 0.$$

That is $\phi_{\tau(y)}(y) \in \Sigma$.

$\square$

The Poincaré map of a periodic orbit

With this result we can define the **Poincaré map** associated with a periodic orbit. Suppose that x is a periodic point with period T, and Σ is a transversal submanifold through x. Then

$$P_\Sigma(y) = \phi_{\tau(y)}(y)$$

is called the Poincaré map. Note that $P_\Sigma : \Sigma \to \Sigma$ and any fixed points of P_Σ are periodic orbits.

This is a nonlinear analogue of the monodromy map from Floquet theory, we will make the relationship more precise soon.

Stability of periodic orbits

If we can show that P_Σ is a contraction of a smaller neighborhood of x_0 then we would obtain stability of the periodic orbit. If $f \in C^1$ the P_Σ is C^1 as well and it would suffice to show that

$$\|DP_\Sigma(x_0)\| < 1.$$

Then $\|DP_\Sigma(y)\| < 1$ in a sufficiently small ball $y \in B_r(x_0)$ and, by fundamental theorem of calculus, P_Σ is a contraction of this neighborhood.

Stability of periodic orbits

If $r > 0$ is chosen as above and

$$\mu := \sup_{B_r(x_0)} \|DP_\Sigma(y)\| < 1$$

then for any $x_1 \in B_r(x_0)$ and $\tau_1 = 0$ the sequence

$$x_n = P_\Sigma(x_{n-1}) \ \text{ and } \ \tau_n = \tau_{n-1} + \tau(x_{n-1})$$

has, by the standard contraction mapping theorem telescoping series argument,

$$|\phi_{\tau_n}(x_1) - x_0| = |x_n - x_0| \leq \frac{\mu^n}{1-\mu}.$$

With a bit more work (homework) you can show that the limit set is $\omega_+(x_1) = \Gamma_{x_0}$.

Relation between Poincaré map and Floquet theory of linearization

Let M be the monodromy map associated with the linearization around a periodic orbit $x(t)$, this equation is also called the **first variational equation**,

$$\dot{z} = Df(x(t))z.$$

Recall $M = \Phi(T)$ where Φ is the principal fundamental solution of linearized equation above and T is the period of the point x_0.

Lemma

Suppose that $f(x_0)$ is normal to Σ at x_0, then the derivative of the Poincaré map is related to the monodromy matrix by

$$DP_\Sigma(x_0) = RM$$

where the matrix R is the orthogonal projection onto the tangent space to Σ at x_0 (i.e. the orthogonal complement of $f(x_0)$).

Poincaré and Floquet

The requirement that $f(x_0)$ is normal to Σ is only to make the formula simpler, in general we have the following formula:

Lemma

The derivative of the Poincaré map is related to the monodromy matrix by

$$DP_\Sigma(x_0) = RM$$

where the matrix R is

$$R = I - \frac{f(x_0)n_{x_0}^T}{f(x_0) \cdot n_{x_0}}.$$

Note that $Rf(x_0) = f(x_0) - f(x_0) = 0$ and $Rv = v$ for v tangent to Σ at x_0.

Poincaré and Floquet

Recall that the monodromy map for $\dot{z} = Df(x(t))z$ always has a unit eigenvalue because $\dot{x}(t)$ is a periodic solution of the linearized equation

$$Mf(x_0) = f(x_0).$$

This degeneracy is nicely removed by the Poincaré map since

$$RMf(x_0) = Rf(x_0) = 0.$$

If 1 appeared as an eigenvalue of M with algebraic/geometric multiplicity 1 then $\mathbb{R}^n = E_s(M) \oplus E_u(M) \oplus \text{span}(f(x_0))$. If Σ is chosen to be tangent to $E_s(M) \oplus E_u(M)$ at x_0 then DP_Σ has the same eigenvectors as M and

$$\sigma(DP_\Sigma) = \sigma(M) \cup \{0\} \setminus \{1\}.$$

Computing derivative of Poincaré map

We compute the derivative of the Poincaré map, blue denotes $n \times 1$ column vectors, and gold denotes $1 \times n$ row vectors,

$$\begin{aligned} DP_\Sigma(y) &= \dot{\phi}_{\tau(y)}(y) D\tau(y) + (D\phi)_{\tau(y)}(y) \\ &= f(P_\Sigma(y)) D\tau(y) + (D\phi)_{\tau(y)}(y). \end{aligned}$$

Evaluating at x_0

$$DP_\Sigma(x_0) = f(x_0) D\tau(x_0) + D\phi_T(x_0).$$

We have two terms to compute, let's start with $D\phi_T(x_0)$.

Computing derivative of Poincaré map

$$DP_\Sigma(x_0) = f(x_0)D\tau(x_0) + D\phi_T(x_0).$$

Note that $D\phi_t(x_0)$ solves the linearized equation

$$\frac{d}{dt}D\phi_t(x_0) = Df(\phi_t(x_0))D\phi_t(x_0) \quad \text{with} \quad D\phi_0(x_0) = I.$$

In fact $D\phi_t(x_0) = \Phi(t)$ is the principal fundamental solution of the linearized equation $\dot{z} = f(\phi_t(x_0))z$ so

$$D\phi_T(x_0) = M$$

is just the monodromy matrix for the linearization.

Derivative of τ

$$DP_\Sigma(x_0) = f(x_0)D\tau(x_0) + D\phi_T(x_0).$$

Let's simplify matters by assuming $\Sigma = \{(x - x_0)\cdot f(x_0) = 0\}$. We compute the derivative $D\tau$ using the implicit equation used to define it

$$f(x_0)^T(\phi_{\tau(y)}(y) - x_0) = 0.$$

Taking the derivative of the equation and evaluating at $y = x_0$ we find

$$f(x_0)^T(\dot\phi_T(x_0)D\tau(x_0) + D\phi_T(x_0)) = 0$$

or

$$f(x_0)D\tau(y) = -\frac{f(x_0)f(x_0)^T}{|f(x_0)|^2}D\phi_T(y) = -\frac{f(x_0)f(x_0)^T}{|f(x_0)|^2}M$$

Stable / unstable manifold theorem

We have already seen in Floquet theory the stable and unstable subspaces associated with a periodic point x_0

$$E_s(M) \text{ and } E_u(M)$$

where M is still the monodromy matrix associated with the linearization $\dot{z} = Df(x(t))z$. We say that the periodic orbit is **hyperbolic** if $\dim(E_s(M) \oplus E_u(M)) = n - 1$.

At the nonlinear level there is a version of the stable/unstable manifold theorem for hyperbolic periodic orbits. The proof is basically the same but the statement is a bit more complicated.

Stable/unstable manifolds

Define the stable manifold for a periodic point x_0

$$M_s(x_0) = \{x| \sup_{t\geq 0} e^{\alpha t}|\phi_t(x) - \phi_t(x_0)| < +\infty \ \text{ for some } \ \alpha > 0\}.$$

Note that for $t_0 \in [0, T)$

$$\phi_{t_0}(M_s(x_0)) = M_s(\phi_{t_0}(x_0))$$

since for $y \in \phi_{t_0}(M_s(x_0))$

$$\begin{aligned} e^{\alpha t}|\phi_t(y) - \phi_t(\phi_{t_0}(x_0))| &= e^{\alpha t}|\phi_t(\phi_{t_0}(x)) - \phi_t(\phi_{t_0}(x_0))| \\ &= e^{\alpha t}|\phi_{t+t_0}(x) - \phi_{t+t_0}(x_0)|. \end{aligned}$$

So the stable set of the entire orbit $\Gamma(x_0)$ would be

$$\begin{aligned} M_s(\Gamma(x_0)) &= \{x| \sup_{t\geq 0} e^{\alpha t} \inf_{y\in\Gamma(x_0)} |\phi_t(x) - y| < +\infty \ \text{ for some } \ \alpha > 0\} \\ &= \cup_{t\in[0,T]}\phi_t(M_s(x_0)) = \cup_{t\in[0,T]} M_s(\phi_t(x_0)). \end{aligned}$$

Visualization

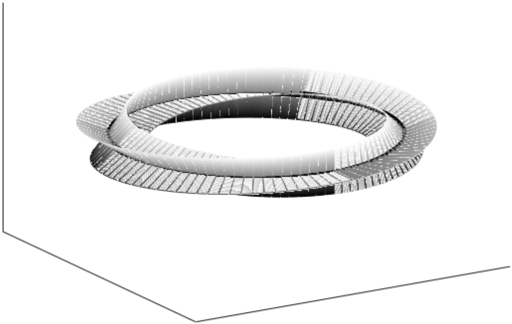


Figure: Stable/unstable manifold visualization. Picture shows case with negative eigenvalues of the monodromy matrix, trajectories orientation with respect to the periodic orbit after one period, and return to original orientation after two periods. Corresponding surfaces are Moëbius strips.

Stable/unstable manifold theorem for periodic orbits

Theorem

Suppose that $f \in C^k$ has a periodic orbit $\Gamma(x_0)$ with corresponding monodromy matrix M hyperbolic. Then there is a neighborhood U of x_0 and a C^k function $h : E_s(M) \cap U \to E_u(M)$ so that

$$M_s(x_0) \cap U = \{y + h(y) : \; y \in E_s(M) \cap U\}.$$

Furthermore both $h(x_0) = x_0$ and $Dh(x_0) = 0$, i.e. $M_s(x_0)$ is tangent to the linearized stable subspace $E_s(M)$ at x_0. The entire stable manifold of the orbit can be written

$$M_s(\Gamma(x_0)) = \{y + h(\phi_{-t}(y)) : \; y \in \phi_t(E_s(M)) \cap \phi_t(U), \; t \in [0, T]\}$$

Hyperbolic periodic orbits persist under small parameter changes

Lemma

Suppose that $f(x,\lambda)$ is C^k in (x,λ) and the flow $\dot{x} = f(x,0)$ has a periodic orbit Γ_{x_0}. Then in a sufficiently small neighborhood U of 0 there is a C^k map $x_0(\lambda)$ so that $\Gamma_{x_0(\lambda)}$ is a periodic orbit for the flow $\dot{x} = f(x,\lambda)$.

Hyperbolic periodic orbits persist under small parameter changes

Proof.

Fix Σ a transversal surface for $f(x,0)$ at x_0, by continuity with respect to λ this surface is also transversal to $f(x,\lambda)$ near x_0 for $|\lambda|$ small. There is a corresponding Poincaré map $P_\Sigma(x,\lambda)$ which is C^k smooth in both variables and so $DP_\Sigma(x,\lambda)$ does not have any eigenvalues with magnitude 1 for small $|\lambda|$ and $|x - x_0|$. Since x_0 is a periodic point for $\lambda = 0$

$$P_\Sigma(x_0, 0) = x_0$$

and

$$D_x(P_\Sigma(x,\lambda) - I) = DP_\Sigma(x,\lambda) - I$$

so since $DP_\Sigma(x,\lambda)$ does not have any eigenvalue 1 this matrix is invertible and implicit function theorem applies. □

Simple limit cycle example

Let's start with an easy example which we have seen before

$$\dot{r} = r(1-r) \quad \text{and} \quad \dot{\theta} = 1$$

which, in (x, y) coordinates, becomes

$$\begin{cases} \dot{x} = x - y - (x^2 + y^2)^{1/2} x \\ \dot{y} = x + y - (x^2 + y^2)^{1/2} y. \end{cases}$$

From the (r, θ) equation it is easier to see that

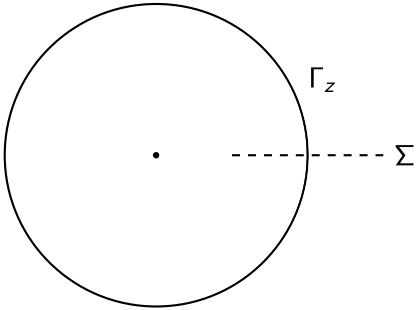
$$z(t) = (\cos(t), \sin(t))$$

is a periodic solution of this system with orbit $\Gamma_z = \{x^2 + y^2 = 1\}$.

Simple limit cycle example

Let's define a Poincaré map for z, we have some freedom in the choice of transversal surface, let's take

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0 \;\text{ and }\; x \in [\tfrac{1}{2}, \tfrac{3}{2}]\}.$$



Simple limit cycle example

From the polar coordinate representation it is clear that for any $u \in \Sigma$ the return time $\tau(u) = 2\pi$. Starting at some point $(x, 0) \in \Sigma$ over that time interval the radial coordinate satisfies

$$\dot{r} = r(1 - r) \ \text{ and } \ r(0) = x$$

so, since that ODE can be solved by separation of variables,

$$r(t) = \frac{e^t}{\frac{1-x}{x} + e^t}.$$

Thus the Poincaré return map is

$$P_\Sigma((x, 0)) = (\frac{e^{2\pi}}{\frac{1-x}{x} + e^{2\pi}}, 0)$$

Simple limit cycle example

Then we can compute

$$\partial_x P_\Sigma((x,0)) = (\frac{e^{2\pi}}{((e^{2\pi}-1)x+1)^2}, 0)$$

and

$$|\partial_x P_\Sigma((1,0))| = \frac{e^{2\pi}}{e^{4\pi}} = e^{-2\pi}.$$

Since this number is strictly smaller than 1 the map P_Σ is a contraction of a sufficiently small neighborhood $\Sigma \cap U$ of $(1,0)$ and the periodic orbit is stable.

Poincaré return map for periodic scalar equations

One interesting special case of the Poincaré map appears in time periodic scalar equations

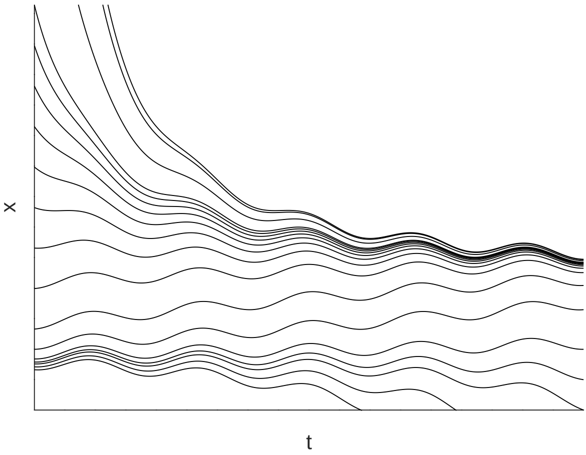
$$\dot{x} = f(t, x)$$

with $f(t + T, x) = f(t, x)$ for some $T > 0$.

The Poincaré map in this setting is defined for all initial data $x_0 \in \mathbb{R}$ and is simply the period mapping

$$Px_0 = \phi_T(x_0)$$

Time periodic scalar equation



Poincaré return map for periodic scalar equations

This is indeed a special case of the previous Poincaré map if we view this scalar equation as a system

$$\dot{t} = 1 \quad \text{and} \quad \dot{x} = f(t, x)$$

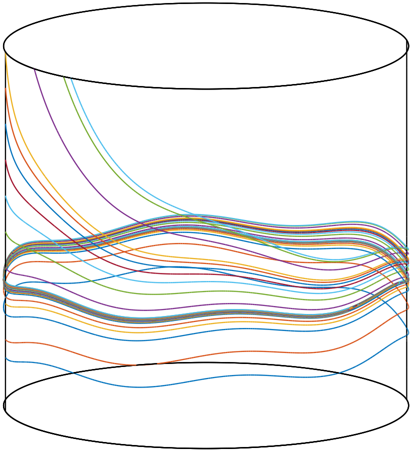
on the cylinder $(t, x) \in (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}$.

The period mapping defined on the previous slide is the Poincaré map for the transversal manifold

$$\Sigma = \{0\} \times \mathbb{R}.$$

Note that the normal to this surface is $(1, 0)$ so the flow field $(1, f(t, x))$ is indeed transversal to Σ.

Wrapping around the cylinder



Application of Poincaré map to a time periodic logistic equation

Let's consider a specific example now

$$\dot{z} = (1 - z)z - h(1 - \sin(2\pi t))$$

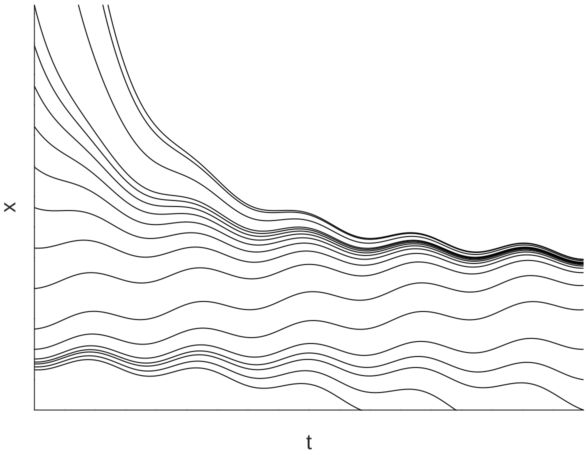
where $h \geq 0$ is a parameter. This is a **logistic growth** equation (population growth with carrying capacity) with a period 1 harvesting term.

We will analyze the Poincaré map

$$P(x) = \phi(1, x).$$

Fixed points of the Poincaré map will correspond to periodic solutions and information at the level of $P'(x)$ will give information on stability of any possible periodic orbits.

Logistic equation with periodic harvesting



Derivative of Poincaré map

Call

$$\theta(t,x) = \partial_x \phi(t,x) \ \text{ and note } \ \theta(t,0) = \partial_x(x) = 1.$$

Then

$$\dot{\theta} = \partial_x \dot{\phi} = (Df)(t,\phi(t,x))\partial_x \phi \ = (1 - 2\phi(t,x))\theta(t,x)$$

and we obtain the formula

$$P'(x) = \theta(1,x) = \exp\left(1 - \int_0^1 2\phi(t,x)\ dt\right).$$

This depends on ϕ itself, but at least we can say that $P'(x) > 0$ so P is strictly increasing (also implied by uniqueness). Actually we also know $\theta(t,x) > 0$ for all t, x.

Second derivative of Poincaré map

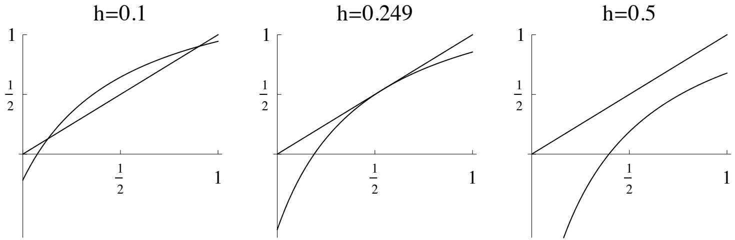
We can differentiate again in the previous expression to find

$$P''(x) = -2\int_0^1 \theta(t,x)\ dxP'(x) < 0.$$

So P is increasing and concave.

This does give us a bit more information because we are looking for solutions of $P(x) = x$ (which would correspond to periodic orbits), and a strictly concave function can cross a line 0, 1 or 2 times.

Crossing possibilities



Note that if the graph of $P(x)$ crosses x exactly once then $P'(x_*) = 1$ at the crossing point, while if P crosses twice then $P'(x_L) > 1$ at the left crossing point and $P'(x_R) < 1$ at the right crossing point. Thus x_L is unstable and x_R is stable.

Figure from Teschl, ODE and Dynamical Systems, page 30

Derivative with respect to h

The previous pictures indicate that we may be able to show existence of periodic orbits by varying h. We try computing the derivative with respect to h now

$$\zeta(t,x,h) = \partial_h \phi(t,x,h) \ \text{ with } \ \zeta(0,x,h) = 0$$

which solves the equation

$$\dot{\zeta} = \partial_\phi f(t,\phi,h)\zeta + \partial_h f(t,\phi,h) = (1-2\phi)\zeta - (1-\sin(2\pi t)).$$

Then, by Duhamel,

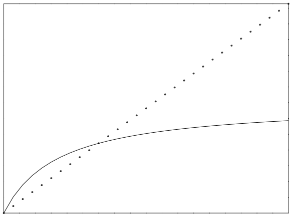
$$\partial_h P(x,h) = \zeta(1,x,h) = -\int_0^1 \exp\left(\int_s^1 1 - 2\phi(t,x)dt\right)(1-\sin(2\pi s))ds$$

which is negative.

Critical harvesting rate

We can explicitly compute when $h = 0$ by the solution of the logistic equation

$$P(x, h = 0) = \frac{ex}{(e-1)x + 1}.$$



Since $P(x, h)$ is continuous with respect to h there are two periodic solutions for small $h > 0$. As h increases $P(x, h)$ decreases and there is a critical value h_c at which the solutions collide and there is only a single periodic solution which is left unstable and right stable.