MATH 6410: Ordinary Differential Equations

Geometric methods for nonlinear systems

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Concepts of Dynamical Systems

What is a dynamical system?

Very broadly a **dynamical system** is the action of a semigroup G with identity element e on a set M. That is a mapping

$$T : G \times M \to M$$

$$(g, x) \mapsto T_g(x)$$

which respects the semi-group property

$$T_g \circ T_h = T_{gh} \ \text{ and } \ T_e = I.$$

- **Discrete dynamical systems** with $G = \mathbb{Z}$ (a group) or $G = \mathbb{N}_0 = \{0, 1, 2, \dots\}$ (a semi-group).
- **Continuous dynamical systems** with $G = \mathbb{R}$ (a group) or $G = [0, \infty)$ (a semi-group).

What is a dynamical system?

Example

- The flow map for an autonomous ODE on $\mathbb{R}^n$ is a continuous time dynamical system on $\mathbb{R}^n$ parametrized by $\mathbb{R}$.
- The application of the monodromy map for a time periodic linear system $\dot{x} = A(t)x$

  $$x_0 \mapsto Mx_0$$

  is a discrete dynamical system parametrized by $\mathbb{Z}$.
- The application of a contraction mapping f on a subset X of a Banach space

  $$x \mapsto f(x)$$

  is a discrete dynamical system parametrized by $\mathbb{N}_0$.

What is a dynamical system?

The concepts we are going to define make sense in this general context. Of course we won't usually be thinking in this generality, but we have already seen examples (the monodromy map) where it can be useful to remember that the concepts used to describe the behaviors of ODE flow maps can also be useful to describe discrete time evolution problems and vice versa.

Reminders

Nonlinear autonomous ODE system on $\mathbb{R}^n$

$$\dot{y} = f(y) \tag{1}$$

with globally Lipschitz continuous f. This assumption is not really necessary for the concepts we will define, just convenient.

The flow map $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is defined

$$\phi_t(x) \ \text{ solves (1) with } \phi_0(x) = x.$$

Reminders

Define **orbits** or **trajectories** of the flow map

$$\Gamma(x) = \cup_{t \in \mathbb{R}} \{\phi_t(x)\}.$$

The **forward** and **backward** trajectories from a point x are

$$\Gamma_{\pm}(x) = \cup_{\pm t > 0} \{\phi_t(x)\}.$$

If $\Gamma(x) = \{x\}$ then x is a **fixed point** or **stationary point** of the flow.

General case

If f is only assumed to be locally Lipschitz, then each point x has a corresponding maximal interval of existence $I_x = (T_-(x), T_+(x))$ and $\phi_t(x)$ is defined for t on this interval. More precisely the flow map is defined on

$$W = \cup_{x \in \mathbb{R}^n} I_x \times \{x\} \subset \mathbb{R} \times \mathbb{R}^n.$$

If $T_+(x) = +\infty$ (resp. $T_-(x) = -\infty$) then the trajectory $\Gamma(x)$ is called $+$-complete (resp. $-$-complete).

We will stick to the case f globally Lipschitz to avoid these details, but the ideas we will define make sense in this more general context of local Lipschitz.

Periodic points

A point $x \in \mathbb{R}^n$ is called a **periodic point** if there is $T > 0$ so that $\phi_T(x) = x$, i.e. if $x \in \Gamma_+(x)$. In this case $\Gamma(x)$ is called a **periodic orbit** or a **closed orbit**.

Lemma

If $\Gamma_+(x) \cap \Gamma_-(x) \neq \emptyset$ then x is a periodic point.

If one point on an orbit $\Gamma(x)$ is periodic then all points on $\Gamma(x)$ are periodic.

Invariant sets

A set $E \subset \mathbb{R}^n$ is called $+$-invariant or positively invariant if

$$\Gamma_+(x) \subset E \quad \text{for all} \quad x \in E.$$

Negatively invariant is defined analogously. A set is called invariant if it is both positively and negatively invariant.

Invariant sets

Lemma

- *Arbitrary intersections of $+$-invariant sets are $+$-invariant.*
- *The closure of a $+$-invariant set is $+$-invariant.*
- *If E and F are invariant then $E \setminus F$ is invariant.*

Invariant set algebra

Proof.

1. If $x \in \cap_\alpha E_\alpha$ all positively invariant then $\Gamma_+(x) \subset E_\alpha$ for each α and so $\Gamma_+(x) \subset \cap_\alpha E_\alpha$.

2. Let E be positively invariant and $x \in \bar{E}$. There is a sequence $E \ni x_n \to x$ and $\Gamma_+(x_n) \subset E$ by invariance. Fix $t > 0$ by continuity of the flow map $\phi_t(x_n) \to \phi_t(x)$ as $n \to \infty$. Thus $\phi_t(x) \in \bar{E}$. Since t was arbitrary $\Gamma_+(x) \subset \bar{E}$.

3. Let $x \in E \setminus F$. Suppose that $y \in \Gamma(x) \cap F$. Then $\Gamma(y) \subset F$, but $\Gamma(y) = \Gamma(x)$ so $x \in F$ which is a contradiction. □

ω limit sets

In looking at the long time behavior of orbits of a dynamical system on $\mathbb{R}^n$ the following idea will be useful:

Definition

The ω_+-limit set of a point $x \in \mathbb{R}^n$ is defined

$$\omega_+(x) = \{y \in \mathbb{R}^n : \exists \ t_n \to \infty \ \text{ such that } \ \lim_{n\to\infty} \phi_{t_n}(x) = y\}.$$

The ω_--limit set is defined analogously with $t_n \to -\infty$.

Examples of ω-limit sets

- (Empty set) Consider the ODE with $a \in \mathbb{R}^n$ nonzero constant

  $$\dot{x} = a$$

  solution are

  $$x(t) = x_0 + at$$

  all solutions go off to infinity and $\omega_\pm(x_0) = \emptyset$ for all x_0.
- (Stable node) Suppose A is a real matrix with all negative eigenvalues

  $$\dot{x} = Ax$$

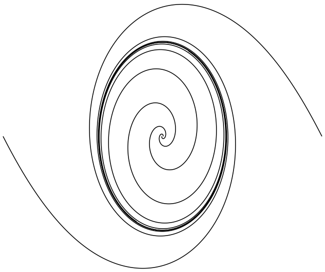
  all solutions have $|x(t)| \to 0$ as $t \to \infty$ and $|x(t)| \to \infty$ as $t \to -\infty$ so

  $$\omega_+(x_0) = \{0\} \ \text{ and } \ \omega_-(x_0) = \emptyset.$$

Examples of ω-limit sets

- (Limit cycle) For example our contrived example from earlier, written in polar coordinates,

$$\dot{r} = r(1-r) \quad \text{and} \quad \dot{\theta} = 1.$$



The limit set is

$$\omega_+(x) = \{r = 1\}$$

for all $x \neq 0$, while

$$\omega_-(x) = \begin{cases} \emptyset & |x| > 1 \\ \{r = 1\} & |x| = 1 \\ \{0\} & |x| < 1. \end{cases}$$

Real system with a limit cycle: Van der Pol Oscillator

Famous example of a system with a limit cycle is **Van der Pol's equation**

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

with $\mu > 1$. Oscillator with nonlinear damping / anti-damping.

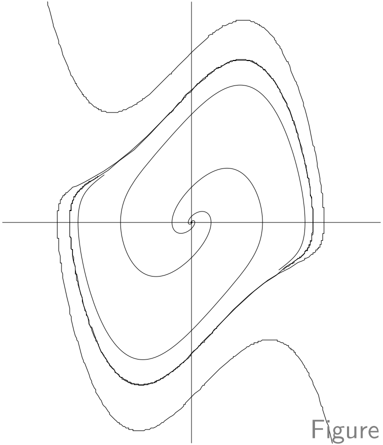


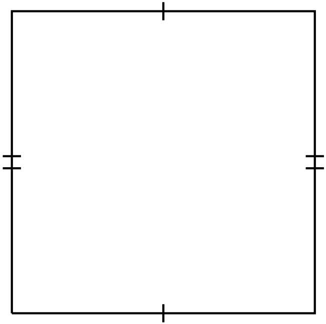
Figure from Teschl, Page 231

Examples of ω-limit sets

Irrational rotations

Limit sets can be *much* more complicated than this (except in $\mathbb{R}^2$). The following is actually a very simple and well-behaved example:

- (Irrational rotation of the torus) Consider the 2-d torus $\mathbb{T}^2 = \mathbb{R}^2 \bmod 2\pi\mathbb{Z}^2$, denote points in $\mathbb{T}^2$ by (θ, ϕ),



Examples of ω-limit sets

Irrational rotations

Can embed in $\mathbb{R}^3$ via the map

$$T(\theta, \phi) = [(2 + \cos\phi)\cos\theta, (2 + \cos\phi)\sin\theta, \sin\phi].$$

So I will describe constructing a flow on $\mathbb{T}^2$, but this can be realized as a flow on $\mathbb{R}^3$ if we want.

Examples of ω-limit sets

Irrational rotations

Consider the flow on $\mathbb{T}^2$

$$\dot{\theta} = 1 \quad \text{and} \quad \dot{\phi} = \alpha$$

with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ irrational. Solution started from $(0,0)$ is

$$\theta(t) = t \text{ mod } 2\pi \quad \text{and} \quad \phi(t) = \alpha t \text{ mod } 2\pi.$$

Solution cannot be periodic with any period T because then $\theta(T) = \phi(T) = 0$ would imply

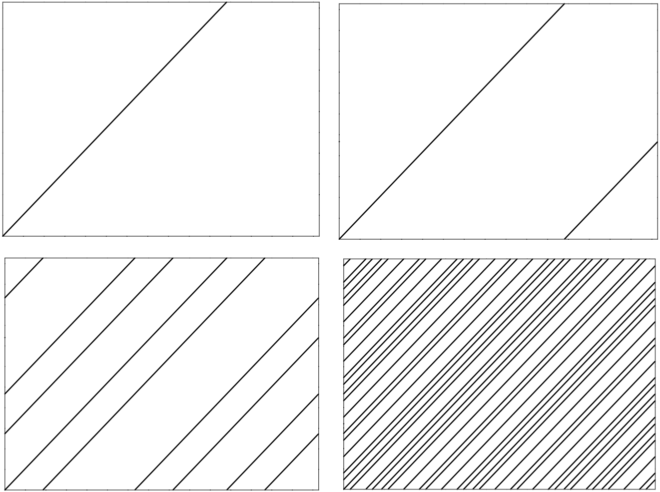
$$T = 2\pi k \quad \text{and} \quad T = \frac{2\pi\ell}{\alpha} \quad \text{for some} \quad k, \ell \in \mathbb{Z}$$

and so $\alpha = \ell/k$ would be rational.

Examples of ω-limit sets

Irrational rotations

More work shows that every orbit is dense in $\mathbb{T}^2$, i.e. $\omega_{\pm}(x) = \mathbb{T}^2$.



ω limit sets - some general properties

With those examples in mind let's discuss some topological properties of limit sets that are true in general.

Lemma

The limit sets $\omega_\pm(x)$ are closed and invariant.

1. (Closed) Let $y_n \in \omega_+(x)$ with $y_n \to y$. Then there are $t_n \to \infty$ such that

$$|\phi_{t_n}(x) - y_n| \leq \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} |\phi_{t_n}(x) - y| \leq \lim_{n\to\infty} [|\phi_{t_n}(x) - y_n| + |y_n - y|] = 0.$$

ω limit sets

1. (Invariant) Let $y \in \omega_+(x)$. There is a sequence $t_n \to \infty$ so that

$$|\phi_{t_n}(x) - y| \to 0.$$

Let $t \in \mathbb{R}$ then

$$|\phi_{t_n+t}(x) - \phi_t(y)| \leq |\phi_t(\phi_{t_n}(x)) - \phi_t(y)| \to 0 \quad \text{as} \quad n \to \infty$$

since ϕ_t is continuous. Thus $\phi_t(y) \in \omega_+(x)$. Since $t \in \mathbb{R}$ was arbitrary $\Gamma(y) \in \omega_+(x)$, and since $y \in \omega_+(x)$ was arbitrary $\omega_+(x)$ is invariant.

Orbits which stay bounded in positive time

Of course the limit sets may be empty if a trajectory goes off to ∞, but if that does not happen we can prove some nice properties:

Lemma

If $\Gamma_+(x)$ is contained in a compact set K then $\omega_+(x)$ is non-empty, compact, and connected.

By the definition of compactness $(\phi_t(x))_{t>0}$ has convergent subsequences so $\omega_+(x)$ is nonempty. Since it is a closed subset of a compact set it is compact.

Orbits which stay bounded in positive time

If $\omega_+(x)$ is disconnected there are two disjoint open sets U and V which cover $\omega_+(x)$ and there are points $y \in U \cap \omega_+(x)$ and $z \in V \cap \omega_+(x)$.

There is an increasing sequence of times $t_n \to \infty$ such that

$$\phi_{t_{2n}}(x) \in U \quad \text{and} \quad \phi_{t_{2n+1}}(x) \in V.$$

Since the map $\phi_t(x)$ is continuous the image $\phi_{[t_n,t_{n+1}]}(x)$ is connected, since it intersects both U and V it must also intersect $K \setminus (U \cup V)$. Thus there is a sequence of times $s_n \in (t_n, t_{n+1})$ such that

$$\phi_{s_n}(x) \in K \setminus (U \cup V).$$

Since that set is compact the sequence $\phi_{s_n}(x)$ has a convergent subsequence converging to a point $w \in K \setminus (U \cup V)$. Then $w \in \omega_+(x) \setminus (U \cup V)$ which is a contradiction.

$\square$

Heteroclinics and Homoclinics

An orbit $\Gamma(x)$ is called a **heteroclinic** orbit between two distinct fixed points y and z if

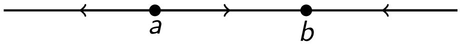
$$\omega_-(x) = \{y\} \text{ and } \omega_+(x) = \{z\}.$$

An orbit is called **homoclinic** if the above holds with $y = z$.

- (Trivial 1-d example) Consider the scalar equation

$$\dot{x} = f(x)$$

with f smooth. Between any subsequent pair of zeros of f there is a heteroclinic orbit (phase line analysis).



Heteroclinics and Homoclinics

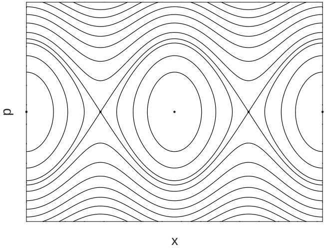
Examples

- (Pendulum model again) Recall the frictionless pendulum model from the beginning of the class

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -\sin x \end{bmatrix}$$

has conserved quantity which allows to draw the phase plane

$$H(p, x) = \frac{1}{2}p^2 + (1 - \cos x).$$



For each k there is a Heteroclinic connecting $(2k-1)\pi$ to $(2k+1)\pi$ and one going the opposite direction.

Heteroclinics and Homoclinics

Examples

- (Homoclinic orbit) Another system from mechanics

$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -U'(x) \end{bmatrix}$$

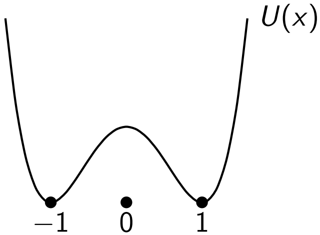
where $U(x) = (1-x^2)^2$, called **double well potential**, and system again has conserved quantity which allows to draw the phase plane

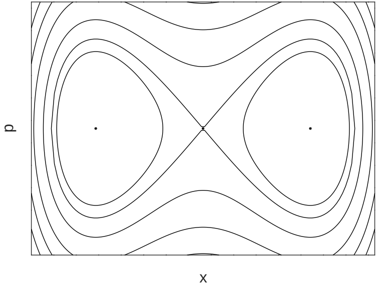
$$H(p,x) = \frac{1}{2}p^2 + U(x).$$

Heteroclinics and Homoclinics

Phase plane analysis of double well system

The double well system has non-degenerate centers at $(-1, 0)$ and $(1, 0)$ and a fixed point at $(0, 0)$ with one unstable and one stable eigenvalue. There are two symmetric homoclinic orbits leaving 0 along an unstable direction and returning along a stable direction.





Can compute homoclinics explicitly by solving $\frac{1}{2}p^2 + (1 - x^2)^2 = 1$.

Heteroclinics and Homoclinics

Phase plane analysis of double well system

Note that

$$Df(x,p) = \begin{bmatrix} 0 & 1 \\ -U''(x) & 0 \end{bmatrix}.$$

If $U''(x)$ is positive, as it is at $x = -1, 1$, this matrix has pure imaginary eigenvalues. While at $x = 0$ we have $U''(0) = -4$, the characteristic polynomial is

$$\lambda^2 - 4 = 0 \ \text{ so } \ \lambda_\pm = \pm 2,$$

and the eigenvectors are

$$v_+ = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ \text{ and } \ v_- = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Connected network of heteroclinics/homoclinics may be an ω - limit set

We can (artificially) modify the previous double well example to make the pair of homoclinics together with the fixed point at $(0,0)$ an ω_+ limit set for orbits started outside. We will just add a small dissipative term which vanishes along the homoclinics:

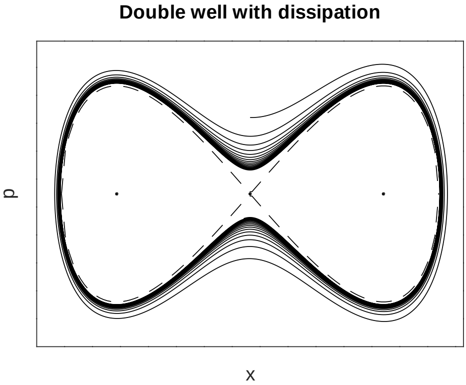
$$\frac{d}{dt}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -\eta(H(p,x)-1)^2 p - U'(x) \end{bmatrix}.$$

For this system the energy $H(p,x)-1$ will be dissipated

$$\begin{aligned}\frac{d}{dt}(H(p(t),x(t))-1) &= p\dot{p} + U'(x)\dot{x} \\ &= p(-\eta(H(p,x)-1)^2 p - U'(x)) + U'(x)p \\ &= -\eta(H(p,x)-1)^2 p^2\end{aligned}$$

but dissipation vanishes as $H(p(t),x(t)) \searrow 1$.

Dissipative double well



Call S_L and S_R the left and right separatrices, L and R the left and right interior regions, and O the outside region.

$\omega_+(x)$	$\omega_-(x)$	$x \in$
$S_L \cup S_R \cup \{0\}$	$\emptyset$	O
$(-1, 0)$	$S_L \cup \{0\}$	L
$(1, 0)$	$S_R \cup \{0\}$	R
0	0	$S_L \cup S_R$

Poincaré-Bendixson Theorem

Although ω-limit sets can be highly complicated in higher dimensions, they are relatively simple in 2-d.

Theorem (Poincaré-Bendixson)

Let U be an open subset of $\mathbb{R}^2$ and $f \in C^1(U, \mathbb{R}^2)$. Fix $x \in U$ and suppose $\omega_+(x) \neq \emptyset$ is compact, connected, and contains only finitely many fixed points. Then ones of the following cases holds:

- *$\omega_+(x)$ is a fixed point.*
- *$\omega_+(x)$ is a periodic orbit.*
- *$\omega_+(x)$ consists of (finitely many) fixed points x_j and non-closed orbits $\Gamma(y)$ such that $\omega_\pm(y) \in \{x_j\}$ (i.e. heteroclinics or homoclinics between the fixed points).*

We cannot prove this immediately but we will work our way to it (with some asides along the way).

Nonlinear stability definitions

Nonlinear (Lyapunov) stability

In the nonlinear setting the meaning of stability becomes a bit more complicated than just computing eigenvalues, we need to make some good definitions to express what we mean.

A fixed point x_0 of a nonlinear system $\dot{x} = f(x)$ is called (Lyapunov) **stable** if, for any neighborhood U of x_0 there is another neighborhood V of x_0 so that any solution started in V stays in U for all $t > 0$.

A fixed point which is not stable is called **unstable**. Or, in more detail, there exists a neighborhood U of x_0 and a sequence $x_n \to x_0$ so that $\Gamma_+(x_n)$ leaves U for all n.

Nonlinear asymptotic stability

A fixed point x_0 of a nonlinear system is called **asymptotically stable** if it is stable and if there is a neighborhood U of x_0 so that

$$\lim_{t\to\infty} |\phi_t(x) - x_0| \quad \text{for all } x \in U.$$

(Note this second condition does not imply stability, so it needs to be added as a separate assumption).

Nonlinear exponential stability

A fixed point x_0 of a nonlinear system is called **exponentially stable** if there are constants $\alpha, \delta, C > 0$ so that

$$|\phi_t(x) - x_0| \leq Ce^{-\alpha t}|x - x_0| \quad \text{for all} \quad |x - x_0| < \delta.$$

(This condition does imply stability).

Linear examples

- Linear system $\dot{x} = Ax$ and all eigenvalues of A have negative real part. System is (globally) exponentially stable.
- Linear system $\dot{x} = Ax$. All eigenvalues of A have non-positive real part and all eigenvalues with zero real part are non-degenerate. This system is stable.

Nonlinear examples

For a nonlinear system

$$\dot{x} = f(x)$$

we have already seen:

- If f is C^1 with a fixed point at x_0 and all eigenvalues of $Df(x_0)$ have negative real part then x_0 is exponentially stable.

If the linearization at the fixed point has some eigenvalues with zero real part then linear information is insufficient to make a conclusion about stability / instability.

- Consider $\dot{x} = \mu x^3$, by phase line analysis the fixed point at 0 is asymptotically stable for $\mu < 0$, stable for $\mu \leq 0$, and unstable for $\mu > 0$.

The Lyapunov method for stability

Lyapunov functions

Let x_0 be a fixed point of a nonlinear system $\dot{x} = f(x)$ and U be a neighborhood of x. A **Lyapunov function** is a continuous function

$$L : U \to \mathbb{R}$$

which is zero at x_0 positive in $U \setminus \{x_0\}$ and non-increasing along the flow

$$L(\phi_t(x)) \leq L(\phi_s(x)) \quad \text{for} \quad t > s \quad \text{and} \quad \phi_{[s,t]}(x) \subset U \setminus \{x_0\}.$$

It is called a **strict** Lyapunov function if strict inequality holds above.

Sub-level invariance

Since L is decreasing along solutions we expect that sublevel sets will be invariant.

For example consider S_δ the connected component of $\{x \in U : L(x) \leq \delta\}$ which contains x_0. We have to be a bit careful because if $\partial S_\delta \cap \partial U \neq \emptyset$ then we cannot guarantee invariance.

Lemma

If S_δ is closed it is positively invariant.

Lyapunov function implies stability

Lemma

If S_δ is closed it is positively invariant.

Lemma

For every $\delta > 0$ S_δ contains a neighborhood of x_0, and for every $r > 0$ there is δ so that

$$S_\delta \subset B_r(x_0).$$

In particular S_δ is closed and invariant for sufficiently small $\delta > 0$.

From these two results we can conclude stability of the fixed point x_0.

Theorem

If x_0 is a fixed point of f such that there exists a Lyapunov function for the flow $\dot{x} = f(x)$ in a neighborhood of x_0 then x_0 is stable.

Proof that closed sub-levels are invariant

Lemma

If S_δ is closed it is positively invariant.

Proof.

Suppose that $x \in S_\delta$ and $\phi_t(x)$ leaves S_δ at time t_0. Since S_δ is closed $y = \phi_{t_0}(x) \in \partial S_\delta \subset U$ and so there is a neighborhood $B_r(y) \subset U$. Then, by assumption, there exist times $t > t_0$ with $\phi_t(x) \in U \setminus S_\delta$ and $\phi_{[t_0,t]}(x) \subset B_r(y) \subset U$ i.e.

$$L(\phi_t(x)) > \delta = L(\phi_{t_0}(x))$$

which contradicts the Lyapunov function property. □

Proof that small δ sublevels converge to x_0

Lemma

For every $\delta > 0$ S_δ contains a neighborhood of x_0, and for every $r > 0$ there is δ so that

$$S_\delta \subset B_r(x_0).$$

In particular S_δ is closed and invariant for sufficiently small $\delta > 0$.

Proof.

Suppose there exists an r so that $S_\delta \not\subset B_r(x_0)$ for all $\delta > 0$. Since S_δ is connected and contains x_0 (by its definition) and also intersects $U \setminus B_r(x_0)$ we must have $S_\delta \cap \partial B_r(x_0) \neq \emptyset$ for all $\delta > 0$. Call y_δ to be a sequence in $S_\delta \cap \partial B_r(x_0)$, by compactness there is a convergent subsequence $y_{\delta_n} \to y_* \in \partial B_r(x_0)$ and

$$L(y) = \lim_{n \to \infty} L(y_{\delta_n}) \leq \lim_{n \to \infty} \delta_n = 0$$

but $L(y_*) = 0$ implies $y_* = x_0$ which is a contradiction. $\square$

Strict Lyapunov function implies asymptotic stability

Note that along any orbit starting at $x \in S_\delta$ (for small enough $\delta > 0$) L is monotone decreasing and so the limit exists

$$L_0(x) = \lim_{t\to\infty} L(\phi_t(x)).$$

So if $y \in \omega_+(x)$, since there is a sequence of t_n so that $\phi_{t_n}(x) \to y$

$$L(y) = \lim_{n\to\infty} L(\phi_{t_n}(x)) = L_0(x).$$

Thus

$$\omega_+(x) \subset \{y : L(y) = L_0(x)\}.$$

This is already a useful piece of information to narrow down the nature of $\omega_+(x)$, and with a bit more we may even conclude that $\omega_+(x) = \{x_0\}$:

Strict Lyapunov function implies asymptotic stability

Theorem

Suppose x_0 *is a fixed point of the flow* $\dot{x} = f(x)$ *and* L *is a Lyapunov function in a neighborhood* U *of* x_0 *such that* L *is not constant on any orbit in* $U \setminus \{x_0\}$ *(e.g. when* L *is a strict Lyapunov function). Then* x_0 *is asymptotically stable and every forward orbit which lies entirely in* U *converges to* x_0*.*

How to find a Lyapunov functional

At this point probably the most mysterious thing is where would you find a Lyapunov functional? It turns out they are fairly common and arise in certain classes of ODE systems, often there is some physical reasoning which can help to figure out possible Lyapunov functions for a system.

We will start looking at several important classes of ODE systems which have Lyapunov functions but first a couple small notes:

- For a differentiable Lyapunov function (which is often the case)
$$0 \geq \frac{d}{dt}L(\phi_t(x)) = L(\phi_t(x)) \cdot f(\phi_t(x))$$
so $L(y) \cdot f(y) \leq 0$ in the region U where L is Lyapunov.
- If L is Lyapunov and $\alpha : \mathbb{R} \to \mathbb{R}$ is monotone then $\alpha \circ L$ is Lyapunov as well. This is useful sometimes for computational reasons, like if $L(y)$ is an absolute value of something $L(y)^2$ may be easier to work with.

Gradient systems / gradient flows

Gradient flows

Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is smooth, consider the ODE

$$\dot{x} = -\mu \nabla V(x).$$

This type of system is called a **gradient flow**. These types of systems arise in the modelling of dissipative phenomena. The flow follows the **direction of steepest descent** of the potential V.

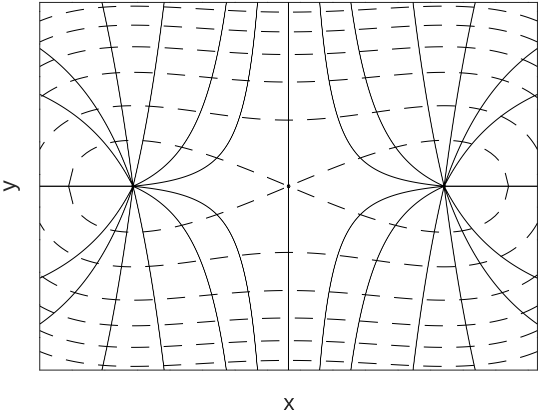
Lemma

A point x is a stationary solution of the gradient flow if $\nabla V(x) = 0$. Suppose that ∇V has only finitely many zeros in any compact set, then strict local minima of V are asymptotically stable fixed points.

Trajectories are orthogonal to the level sets of V

For example take the two-dimensional double well potential

$$V(x,y) = \frac{1}{2}y^2 + \frac{1}{8}(1-x^2)^2$$



Energy dissipation inequality

The fundamental inequality satisfied by gradient systems comes from taking the time derivative of the potential

$$\frac{d}{dt}V(x) = \dot{x}(t) \cdot \nabla V(x(t)) = -\mu |\nabla V(x)|^2 < 0.$$

Which we rewrite for emphasis

$$\frac{d}{dt}V(x) = -\mu |\nabla V(x)|^2.$$

In particular, $V(x)$ is a Lyapunov functional for its gradient flow! It is a strict Lyapunov functional in any neighborhood of a fixed point x_0 which contains no other fixed points.

Strict local minima of the potential

Lemma

A point x is a stationary solution of the gradient flow if $\nabla V(x) = 0$. Suppose that ∇V has only finitely many zeros in any compact set, then strict local minima of V are asymptotically stable fixed points.

Proof.

Given a strict local minimum x_0 of V, choose a neighborhood U so that $\{\nabla V = 0\} \cap U = \{x_0\}$. Then V is a strict Lyapunov functional in U for the flow and the previous general results apply. □

Hamiltonian systems

Hamiltonian mechanics

A **Hamiltonian** $H(p,x)$ is a C^1 real valued function on $(p,x) \in \mathbb{R}^n \times \mathbb{R}^n$. Hamiltonians of classical mechanics typically take the form

$$H(p,x) = \frac{1}{2m}|p|^2 + V(x)$$

which is the sum of **kinetic** and **potential** energy, but we do not need to restrict ourselves to this form.

The **Hamiltonian system** is the following ODE associated with the Hamiltonian H

$$\begin{cases} \dot{x} = D_p H(p,x) \\ \dot{p} = -D_x H(p,x). \end{cases}$$

Energy conservation

Hamiltonian systems essentially behave in a way which is orthogonal to gradient systems, trajectories are *parallel* to the level sets of the energy $H(p,x)$. This leads to **energy conservation** along solution trajectories

$$\begin{aligned}\frac{d}{dt}H(p,x) &= D_pH(p,x)\cdot\dot{p} + D_xH(p,x)\cdot\dot{x}\\ &= -D_pH(p,x)\cdot D_xH(p,x) + D_xH(p,x)\cdot D_pH(p,x)\\ &= 0.\end{aligned}$$

The Hamiltonian $H(p,x)$ is a (definitely non-strict) Lyapunov functional for the associated Hamiltonian system!

Stationary solutions

Note that when has the Newtonian mechanics form

$$H(p,x) = \frac{1}{2m}|p|^2 + V(x)$$

the fixed points of the Hamiltonian system occur where

$$p = D_p H(p,x) = 0 \ \text{ and } \ \nabla V(x) = D_x H(p,x) = 0.$$

Note that $H(p,x)$ has a local minimum where V has a strict local minimum, and it has a saddle where V has a strict local maximum.

Strict local minima of the Hamiltonian are stable

Lemma

If (p_0, x_0) is a strict local minimum of H then it is (Lyapunov) stable.

Proof.

By the set up there is a neighborhood U of (p_0, x_0) so that $H(p, x) > H(p_0, x_0)$ for $(p, x) \in U$. Thus H is a Lyapunov function for the flow in U and so (p_0, x_0) is stable. $\square$

More properties of Hamiltonian systems

Theorem (Liouville)

The Hamiltonian flow preserves volumes in phase space $(p, x) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof.

Recall that divergence free flows are volume preserving and

$$\mathrm{div}_{p,x}([-\nabla_x H(p,x), \nabla_p H(p,x)]) = -\nabla_p \cdot \nabla_x H + \nabla_x \cdot \nabla_p H = 0$$

$\square$

Another conclusion we can draw from this computation: the trace of the linearization at any critical point is zero, so the Hamiltonian system cannot have any exponentially stable fixed points either forward or backwards in tie.

Adding dissipation to a Hamiltonian system

Mechanical systems with friction often give rise to a system which is Hamiltonian with an additional term which we call **dissipative** in the momentum equation

$$\begin{cases} \dot{x} = D_p H(p,x) \\ \dot{p} = -\mu D_p H(p,x) - D_x H(p,x). \end{cases}$$

with $\mu \geq 0$. One could also allow $\mu(p,x) \geq 0$. Note that this additional term does not change the fixed points of the system.

If $H(p,x) = \frac{1}{2}|p|^2 + V(x)$, as is the case for Newton's equations in a potential, then the dissipative term usually has the form $-\mu p$, although nonlinear dissipation is possible as well.

Energy dissipation

Energy is no longer conserved for this system but we can still show a **dissipation inequality** which is useful

$$\begin{aligned}\frac{d}{dt}H(p,x) &= D_pH(p,x)\cdot\dot{p} + D_xH(p,x)\cdot\dot{x} \\ &= -\mu|D_pH(p,x)|^2 \leq 0.\end{aligned}$$

Thus H can still serve as a Lyapunov functional for the flow near local minima.

Strict local minima of the potential are asymptotically stable

Lemma

If $H(p,x) = \frac{1}{2}|p|^2 + V(x)$ has only finitely many critical points in any bounded set, $\mu > 0$ and (p_0, x_0) is a strict local minimum of H then (p_0, x_0) is asymptotically stable for the dissipative flow.

Strict local minima of the potential are asymptotically stable

Proof.

By the set up there is a neighborhood U of (p_0, x_0) so that $H(p, x) > H(p_0, x_0)$ for $(p, x) \in U$ and (p_0, x_0) is the only fixed point of the flow in U. Thus H is a Lyapunov function for the flow in U and so (p_0, x_0) is stable.

We just need to show that $H(p, x)$ cannot be constant on any orbit $\Gamma_+(p_1, x_1)$ which lies completely in $U \setminus \{(p_0, x_0)\}$. By the energy dissipation equation if H is constant on $\Gamma_+(p_1, x_1)$ then $|D_p H(p_1, x_1)| = |p| = 0$ which means $x(t) = x_1$ is constant as well. Thus (p_1, x_1) is a stationary solution in $U \setminus \{(p_0, x_0)\}$ which contradicts the choice of U. $\square$

Gradient systems as the large dissipation limit of Newtonian mechanics

Consider the dissipative system from mechanics

$$\begin{cases} \dot{x} = p \\ \dot{p} = -\mu p - \nabla V(x) \end{cases}$$

with very large $\mu \gg 1$. As $\mu \to \infty$ this system converges to a gradient flow of the potential V in an appropriate rescaling.

Theorem

Suppose that $V(x) \to +\infty$ as $|x| \to \infty$. For every initial data (x_0, p_0) call (x^μ, p^μ) to be the solution of the above system, then the limit $z(t) = \lim_{\mu\to\infty} x^\mu(\mu t)$ exists and solves

$$\dot{z} = -\nabla V(z) \;\; \textit{with} \;\; z(0) = x_0 + p_0.$$

Finding the right scaling

Writing

$$\dot{x} = p = -\frac{1}{\mu}\nabla V(x) - \frac{1}{\mu}\dot{p}$$

we are lead to guess that x and p are moving on the time scale μ and the length scale of p is $\frac{1}{\mu}$.

We define new variables at these respective scales

$$y(t) = x(\mu t) \ \text{ and } \ q(t) = \mu p(\mu t)$$

and get a rescaled system

$$\begin{cases} \dot{y} = q \\ \dot{q} = -\mu(q + \nabla V(y)) \end{cases} \qquad \text{with} \ \ (y_0, q_0) = (x_0, \mu p_0).$$

Analyzing the momentum equation

We integrate the momentum equation using Duhamel

$$q(t) = e^{-\mu t}\mu p_0 - \int_0^t e^{-\mu(t-s)}\mu \nabla V(y(s))\, ds.$$

We will be guided by the following line of intuition in this proof

1. q remains bounded so that
2. $|\dot{y}| = |q| \sim 1$ while $|\dot{q}| \sim \mu$ so
3. we can treat $y(s)$ as a constant in the Duhamel formula for q.

In reality it is slightly more complicated than this due to the initial time layer of order $1/\mu$ where q decays from μp_0 to unit size.

Upper bound of q

Since (p, x) has to lie in the sublevel $K = \{H(p, x) \leq H(p_0, x_0)\}$ (compact) we can bound

$$|\nabla V(y(t))| = |\nabla V(x(\mu t))| \leq Q(p_0, x_0) := \sup\{|\nabla V(x)| : \Pi_x K\}.$$

So now we can use Duhamel to bound q

$$\begin{aligned} |q(t)| &\leq e^{-\mu t}\mu|p_0| + \int_0^t e^{-\mu(t-s)}\mu|\nabla V(y(s))| \, ds \\ &= e^{-\mu t}\mu|p_0| + Q \end{aligned}$$

and so for any $s < t$

$$|y(t) - y(s)| \leq \int_s^t |q(u)| du \leq (Q + \mu|p_0|e^{-\mu s})(t - s).$$

Approximate value of q

Now let's try to compute the precise value of q using Duhamel again and thinking of y as approximately constant:

$$
\begin{aligned}
q(t) &= e^{-\mu t}\mu p_0 - \int_0^t e^{-\mu(t-s)}\mu \nabla V(y(s))\, ds \\
&= -\int_0^t e^{-\mu(t-s)}\mu \nabla V(y(t))\, ds + e^{-\mu t}\mu p_0 \\
&\qquad \cdots + \int_0^t e^{-\mu(t-s)}\mu[\nabla V(y(t)) - \nabla V(y(s))]\, ds \\
&= -\nabla V(y(t)) + e^{-\mu t}\mu p_0 + B(t)
\end{aligned}
$$

where the error term is

$$
B(t) = e^{-\mu t}\nabla V(y(t)) + \int_0^t e^{-\mu(t-s)}\mu[\nabla V(y(t)) - \nabla V(y(s))]\, ds
$$

Error term estimation

For the second term in $B(t)$ we use the estimate of $|y(t) - y(s)|$ that we established earlier

$$|\nabla V(y(t)) - \nabla V(y(s))| \leq M(Q + \mu |p_0| e^{-\mu s})(t - s)$$

where

$$M = \sup\{\|D^2 V(x)\|_{op} : x \in \Pi_x \text{hull}(K)\}$$

Then we can compute

$$\int_0^t e^{-\mu(t-s)} \mu (t-s)\, ds = \frac{1}{\mu} \int_0^{\mu t} e^{-u} u\, du \leq \frac{1}{\mu} \int_0^{\infty} e^{-u} u\, du = \frac{1}{\mu}$$

and

$$\int_0^t e^{-\mu(t-s)} \mu^2 e^{-\mu s} (t-s)\, ds = \frac{1}{2} t^2 \mu^2 e^{-\mu t}$$

Error term estimation

Combining the previous estimates,

$$|B(t)| \leq Qe^{-\mu t} + M(Q\frac{1}{\mu} + \frac{1}{2}t^2\mu^2|p_0|e^{-\mu t})$$

and using that

$$\int_0^t t^2\mu^2 e^{-\mu t}\ dt = \frac{1}{\mu}\int_0^{\mu t} e^{-u}u^2\ du \leq \frac{1}{\mu}\int_0^{\infty} e^{-u}u^2\ du = \frac{2}{\mu}$$

we find

$$\int_0^t |B(s)|\ ds \leq [Q + M(Qt + |p_0|)]\frac{1}{\mu} = C(t)\frac{1}{\mu}$$

Initial data term

Last we compute the effect of the initial data term

$$\int_0^t e^{-\mu s} \mu p_0 \, ds = p_0(1 - e^{-\mu t}).$$

Approximate integral equation for y

Finally we can write down the approximate integral equation for y

$$
\begin{aligned}
y(t) &= y_0 + \int_0^t q(s)\ ds \\
&= y_0 - \int_0^t \nabla V(y(s))\ ds + \int_0^t e^{-\mu t}\mu p_0\ dt + \int_0^t B(t)\ dt
\end{aligned}
$$

or

$$
\left| y(t) - \left(y_0 + p_0 + \int_0^t -\nabla V(y(s))\ ds \right) \right| \leq C(t)\frac{1}{\mu} + |p_0|e^{-\mu t}.
$$

Error estimate between y and the gradient flow solution z

Now if z is the actual solution of $\dot{z} = -\nabla V(z)$ with initial data $z(0) = x_0 + p_0 = y_0 + p_0$ then

$$|y(t) - z(t)| \leq C(t)\frac{1}{\mu} + |p_0|e^{-\mu t} + \int_0^t M|y(s) - z(s)|\ ds$$

so an application of Grönwall inequality on some fixed time interval $[0, T]$ gives

$$\begin{aligned}|y(t) - z(t)| &\leq C(T)\frac{1}{\mu} + |p_0|e^{-\mu t} + \int_0^t e^{M(t-s)}M(C(T)\frac{1}{\mu} + |p_0|e^{-\mu s})\ d \\ &\leq C(T)(1+M)e^{MT}\frac{1}{\mu} + |p_0|e^{-\mu t} + \frac{1}{M+\mu}|p_0|e^{MT}\end{aligned}$$

for all $t \in [0, T]$. All terms vanish as $\mu \to \infty$.

Gradient systems as the large dissipation limit of Newtonian mechanics

Since it has been a long proof let's just recall what we have established: we were considering the dissipative Newtonian system

$$\begin{cases} \dot{x} = p \\ \dot{p} = -\mu p - \nabla V(x) \end{cases}$$

with very large $\mu \gg 1$. As $\mu \to \infty$ this system converges to a gradient flow of the potential V in an appropriate rescaling.

Theorem

Suppose that $V(x) \to +\infty$ as $|x| \to +\infty$. Then for every initial data (x_0, p_0) call (x^μ, p^μ) to be the solution of the above system, the limit $z(t) = \lim_{\mu \to \infty} x^\mu(\mu t)$ exists and solves

$$\dot{z} = -\nabla V(z) \ \ \textit{with} \ \ z(0) = x_0 + p_0.$$

More on Hamiltonian systems etc

There is significantly more to be said about Hamiltonian systems (and about gradient systems), we may come back to the topic later.

Attracting sets

Stable and unstable sets

Stable and unstable sets

Given an invariant set $E \subset \mathbb{R}^n$ define the (global) **stable set** of E

$$W_s(E) = \{x \in \mathbb{R}^n : \lim_{t \to \infty} d(\phi_t(x), E) = 0\}.$$

The unstable set $W_u(E)$ is defined with time reversed. Note that these sets are both invariant, and therefore their boundaries $\partial W_s(E)$ and $\partial W_u(E)$ are invariant as well.

(Recall that the closure of an invariant set is invariant and the difference of invariant sets is invariant).

Examples of stable and unstable sets

- Global stable/unstable manifold of a hyperbolic stationary point.
- Global stable/unstable manifold of a hyperbolic periodic orbit.

Attracting sets

A closed forward invariant set A is called an **attracting set** if the stable set of A, $W_s(A)$, contains an open neighborhood of A. In that case $W_s(A)$ is called the **domain of attraction** for A or the **basin of attraction** for A.

If U is any positively invariant neighborhood of A, then

$$W_s(A) = \cup_{t<0} \phi_t(U).$$

In particular, when A is attracting, the basin of attraction $W_s(A)$ is open.

Examples of attracting sets

- $$\begin{cases} \dot{x} = x(1 - x^2) \\ \dot{y} = -y \end{cases}$$

  "Interesting" attracting sets are $(-1, 0)$, $(1, 0)$ and $[-1, 1] \times \{0\}$.
- Duffing's equation

  $$\ddot{x} + \delta\dot{x} - x(1 - x^2) = 0$$

  "Interesting" attracting sets are $(-1, 0)$ and $(1, 0)$.
- System with limit cycle (e.g. van der Pol oscillator). Limit cycle is attracting.

How to find an attracting set

An open, bounded, and connected region E is called a **trapping region** for the flow if

$$\phi_t(\overline{E}) \subset E$$

for all $t > 0$.

For example if E also has smooth boundary with outward normal n_x then the region is trapping if

$$f(x) \cdot n_x < 0 \quad \text{for all} \quad x \in \partial E.$$

This condition is straightforward to check computationally especially if the region has a level set representation $E = \{S(x) < 0\}$ because then $n_x = \nabla S(x)$.

How to find an attracting set

Lemma

Let E be a trapping set then

$$\Lambda = \omega_+(E) = \bigcap_{t \geq 0} \phi_t(E)$$

is a nonempty, invariant, compact and connected attracting set.

This may not give the “correct” attracting set.

Lemma

Let E be a trapping region. Then

$$W_u(x) \subset \omega_+(E)$$

for every fixed point $x \in \omega_+(E)$.

Proof.

For any $y \in W_u(x)$ we know $\lim_{t \to -\infty} \phi_t(y) = x \in E$ open so there is some $t_0 < 0$ so that $\phi_{t_0}(y) \in E$ but then by forward invariance of E we know $y \in \Gamma_+(\phi_{t_0}(y)) \subset E$ and sending $t_0 \to -\infty$ we even find the entire orbit $\Gamma(y) \subset E$. But then $\Gamma(y) \in \cap_{t>0} \phi_t(E)$. □

Attractors

To avoid this "degeneracy" in the definition of attracting set we want some notion of a "minimal" attracting set. We call a closed invariant set Λ to be **topologically transitive** if for any two (relatively) open sets $U, V \subset \Lambda$ there is some t so that $\phi_t(U) \cap V \neq \emptyset$. A topologically transitive attracting set is called an **attractor**. In particular an attractor cannot be split into smaller attracting sets.