MATH 6410: Ordinary Differential Equations

Linear systems

Instructor: Will Feldman

University of Utah

Constant coefficient linear systems

Linear systems

In this section we will study linear systems of ODE

$$\dot{x} = Ax \quad \text{with} \quad x(0) = x_0.$$

Here $x(t) \in \mathbb{R}^n$ and A is an $n \times n$ matrix.

Besides their independent importance linear systems arise when one considers solutions of a nonlinear autonomous system near a critical point. We will make this precise in various ways as we proceed in the course.

One motivation: linearizing near a critical point

For intuition think of the following (only formal) calculation:
Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, $f(0) = 0$ and consider the nonlinear system

$$\dot{x} = f(x) \ \text{ and } \ x(0) = \varepsilon y_0$$

with small initial data. Write $x(t) = \varepsilon y(t)$ and rewrite

$$\dot{y} = \frac{1}{\varepsilon} f(\varepsilon y) = \frac{1}{\varepsilon}(f(0) + \varepsilon Df(0)y + O(\varepsilon^2|y|^2))$$

using that 0 is a critical point and ignoring terms of order ε and higher we find

$$\dot{y} \approx Df(0)y \ \text{ and } \ y(0) = y_0.$$

Matrix exponential

Definition

If A is an $n \times n$ matrix we define the **matrix exponential**

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This can be difficult to compute, we will discuss how to do it, but at least we can first show that the infinite sum converges.

Matrix exponential

Recall the operator norm of a matrix

$$\|A\|_{op} = \sup_{x \neq 0} \frac{|Ax|}{|x|}.$$

Which has the nice property

$$\|AB\|_{op} \leq \|A\|_{op}\|B\|_{op}$$

We just need to show that the series is absolutely summable in the operator norm. We compute

$$\|\sum_{k=0}^{\infty} \frac{A^k}{k!}\|_{op} \leq \sum_{k=0}^{\infty} \frac{\|A^k\|_{op}}{k!} \leq \frac{\|A\|_{op}^k}{k!} = e^{\|A\|_{op}}$$

Product formula

Note that, in general, it is not necessarily the case that

$$e^A e^B = e^{A+B}.$$

However, this does hold when A and B commute. This can be verified simply by taking the product of the series representations and using the commutation. One particular consequence:

Lemma

If A is an $n \times n$ matrix then e^A is invertible and its inverse is e^{-A}.

Solution matrix of a linear system

The matrix exponential e^{At} is itself a **matrix solution** of linear system

$$\dot{X} = AX \quad \text{and} \quad X(0) = I.$$

We compute formally

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\sum_{k=0}^{\infty}\frac{A^k t^k}{k!} = \sum_{k=1}^{\infty}\frac{A^k t^{k-1}}{(k-1)!} = Ae^{At}.$$

This requires pulling a derivative inside an infinite sum, but is justified by previous arguments which show the uniform convergence of both the original sum and the sum for the derivative.

Flow map

This actually explicitly identifies the flow map for the ODE $\dot{x} = Ax$:

$$\phi_t(x_0) = e^{At}x_0$$

is a solution of

$$\frac{d}{dt}\phi_t(x_0) = A\phi_t(x_0) \quad \text{and} \quad \phi_0(x_0) = x_0.$$

So the study of linear constant coefficient ODE initial value problems reduces to understanding matrix exponentials.

Variation of parameters / Duhamel's formula

If we add an inhomogeneous term we can still easily determine the flow map. Consider a system of the form

$$\dot{x} = Ax + f(t) \quad \text{with} \quad x(0) = x_0.$$

The solution is given by the **Variation of parameters** or **Duhamel** formula

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s)\, ds.$$

Proof / derivation of Duhamel

Multiply both sides of the equation

$$\dot{x} = Ax + f(t)$$

on the left by e^{-At} and rearrange to find

$$e^{-At}\dot{x} - Ae^{-At}x = e^{-At}f(t).$$

Note that the left hand side is the time derivative of $e^{-At}x(t)$.

Then integrate from time 0 to time t

$$e^{-At}x(t) - x_0 = \int_0^t e^{-As}f(s)\,ds.$$

Then multiply both sides on the left by e^{At} and note that $e^{At}e^{-As} = e^{A(t-s)}$ because A commutes with itself.

Diagonal systems

The easiest case to compute the matrix exponential / solve the linear ODE is when the matrix in question is diagonal, i.e. the system is uncoupled

$$\dot{x} = \Lambda x \quad \text{with} \quad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

In that case

$$\Lambda^k = \operatorname{diag}(\lambda_1^k, \ldots, \lambda_n^k)$$

and so

$$e^{\Lambda t} = \operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}).$$

Allowing complex matrices / solutions can immediately derive that all solutions stay bounded in positive times if and only if $\mathrm{Re}(\lambda_j) \leq 0$ for all $1 \leq j \leq n$ (such statement about eigenvalues of general matrix will more info when some eigenvalues have zero real part)

Diagonal systems

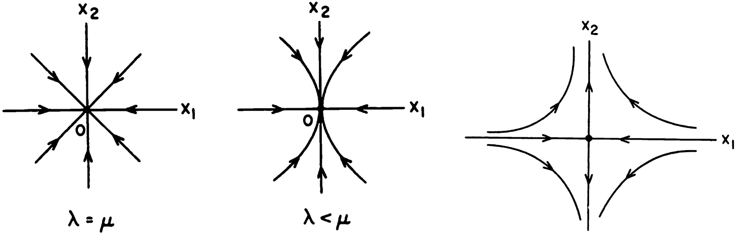


Figure: $\max \lambda_j < 0$: stable node, $\min \lambda_j > 0$: unstable node (reverse arrows), $\min \lambda_j < 0 < \max \lambda_j$: saddle

figures from Perko, Chapter 1.5

Real distinct eigenvalues

The next easiest case is when the matrix A has n distinct real eigenvalues. Then there are n linearly independent eigenvectors $v_1, \dots, v_n$ with $Av_j = \lambda_j v_j$. Call $Q = [v_1, \dots, v_n]$ then we can change coordinates to a diagonal matrix

$$AQ = [Av_1, \cdots, Av_n] = [\lambda_1 v_1, \dots, \lambda_n v_n] = \Lambda Q$$

where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Since Q is invertible we have $Q^{-1}AQ = \Lambda$.

Real distinct eigenvalues

We can use the coordinate transformation to compute the matrix exponential. First note that

$$(Q^{-1}AQ)^k = Q^{-1}AQQ^{-1}AQ\dots QQ^{-1}AQ = Q^{-1}A^kQ.$$

Plugging this formula into the matrix exponential formula

$$e^{\Lambda t} = e^{Q^{-1}AQt} = \sum_{k=0}^{\infty} \frac{(Q^{-1}AQ)^k t^k}{k!} = Q^{-1}(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!})Q = Q^{-1}e^{At}Q$$

The flow map of the linear system is a coordinate change of the flow map for a diagonal system.

Complex eigenvalues

Since A is a real matrix complex eigenvalues must come in complex conjugate pairs $\lambda_\pm = \rho \pm i\omega$ with corresponding complex conjugate pair eigenvectors $v, \bar{v}$.

Let's consider the 2×2 case just to get started, change coordinates via $Q = [\mathrm{Re}(v), \mathrm{Im}(v)]$

$$\begin{aligned} AQ &= [\mathrm{Re}(Av), \mathrm{Im}(Av)] \\ &= [\mathrm{Re}(\lambda v), \mathrm{Im}(\lambda v)] \\ &= [\rho \mathrm{Re}(v) - \omega \mathrm{Im}(v), \omega \mathrm{Re}(v) + \rho \mathrm{Im}(v)] \\ &= \left[\begin{array}{cc} \rho & -\omega \\ \omega & \rho \end{array} \right] Q. \end{aligned}$$

Complex eigenvalues

This is called the real canonical form of the matrix

$$Q^{-1}AQ = \begin{bmatrix} \rho & -\omega \\ \omega & \rho \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = R + \Omega.$$

The diagonal matrix $R = \rho I$ causes exponential growth / decay $e^{\rho t}$, the skew symmetric matrix Ω causes rotation with period $\frac{2\pi}{\omega}$. By same arguments from before

$$e^{Q^{-1}AQt} = Q^{-1}e^{At}Q$$

so we just need to compute the matrix exponential of the canonical form, and since Ω and R commute $e^{(R+\Omega)t} = e^{Rt}e^{\Omega t}$.

Complex eigenvalues

For skew symmetric matrix Ω we compute by induction the powers are

$$\Omega^{2n} = (-1)^n \begin{bmatrix} \omega^{2n} & 0 \\ 0 & \omega^{2n} \end{bmatrix}$$

and

$$\Omega^{2n+1} = (-1)^n \begin{bmatrix} 0 & -\omega^{2n+1} \\ \omega^{2n+1} & 0 \end{bmatrix}$$

so

$$\begin{aligned} e^{\Omega t} &= \begin{bmatrix} \sum_{n=0}^{\infty}(-1)^n \frac{\omega^{2n}t^{2n}}{(2n)!} & -\sum_{n=0}^{\infty}(-1)^n \frac{\omega^{2n+1}t^{2n+1}}{(2n+1)!} \\ \sum_{n=0}^{\infty}(-1)^n \frac{\omega^{2n+1}t^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty}(-1)^n \frac{\omega^{2n}t^{2n}}{(2n)!} \end{bmatrix} \\ &= \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix}. \end{aligned}$$

Complex eigenvalues

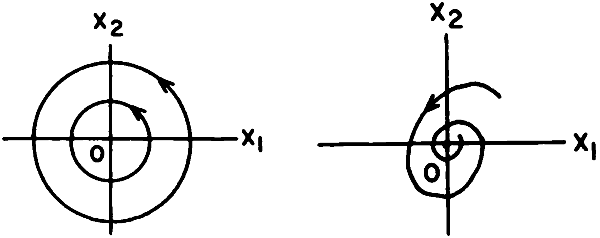


Figure: $\rho = 0$: center, $\rho < 0$: stable spiral, $\rho > 0$ unstable spiral (reverse arrows). Sign of ω determines rotation direction $\omega > 0$ counter-clockwise, $\omega < 0$ clockwise.

figures from Perko, Chapter 1.5

Distinct eigenvalues

If the matrix A has n distinct eigenvalues (real or complex) then we can combine the previous two ideas and put the matrix in normal form.

List the eigenvalues

$$\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \bar{\lambda}_{k+1} \ldots, \lambda_{k+\ell}, \bar{\lambda}_{k+\ell}$$

all distinct with $\lambda_j = \rho_j + i\omega_j$, the first k real and the last 2ℓ complex. Each eigenvalue has an associated eigenvector $w_j = u_j + iv_j$ together forming a basis of $\mathbb{R}^n$

$$B = \{u_1, \ldots, u_k, u_{k+1}, v_{k+1}, \ldots, u_\ell, v_\ell\}.$$

Distinct eigenvalues

Then the real normal form can be written as a **block diagonal matrix**

$$Q^{-1}AQ = \mathrm{diag}(\lambda_1, \dots, \lambda_k, D_1, \dots, D_\ell)$$

with the coordinate transformation

$$Q = [u_1, \dots, u_k, u_{k+1}, v_{k+1}, \dots, u_{k+\ell}, v_{k+\ell}]$$

and block matrices

$$D_j = \begin{bmatrix} \rho_j & -\omega_j \\ \omega_j & \rho_j \end{bmatrix}.$$

Degenerate eigenvalues

Recall that eigenvalues are roots of the characteristic polynomial

$$0 = \det(A - \lambda I) = \prod_{j=1}^{n} (\lambda - \lambda_j).$$

The **algebraic multiplicity** $\alpha(\mu)$ of an eigenvalue μ is the order of the root of the characteristic polynomial at μ.

On the other hand each eigenvalue has at least one eigenvector and we can define the eigenspace

$$\ker(A - \mu I) = \{v \in \mathbb{C}^n : (A - \mu I)v = 0\}.$$

The dimension of the eigenspace associated with μ is always less than or equal to the algebraic multiplicity of the eigenvalue μ and it is called the **geometric multiplicity**.

Generalized eigenspaces

When the algebraic multiplicity is strictly larger than the geometric multiplicity we need to look at the **generalized eigenspaces**

$$E_k(\mu) = \{v \in \mathbb{C}^n : (A - \mu I)^k v = 0\}.$$

The generalized eigenspaces have a triangular structure

$$(A - \mu I)E_k(\mu) \subset E_{k-1}(\mu).$$

Cyclic vectors

It turns out we can reduce to the case of a single generalized eigenspace $\mathbb{R}^k = E_k(\mu)$ with a **cyclic vector** v so that

$$\mathrm{span}(v, Av, \cdots, A^{k-1}v) = \mathbb{R}^k$$

and changing coordinates with

$$Q = [v, (A-\mu I)v, \cdots, (A-\mu I)^{k-1}v]$$

we get

$$Q^{-1}AQ = \begin{bmatrix} \mu & 1 & 0 & \cdots & 0 \\ 0 & \mu & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \mu & 1 \\ 0 & \cdots & \cdots & 0 & \mu \end{bmatrix}.$$

Degenerate eigenvalues: simplest example

The simplest example of a degenerate eigenvalue is the matrix

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

In this case the matrix cannot be diagonalized by a coordinate transform, but we can still compute the matrix exponential. In particular note that the matrix

$$N = A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is **nilpotent**, $N^m = 0$ for some power m (in this case $m = 2$ works).

Degenerate eigenvalues: simplest example

The matrix exponential can then be computed by

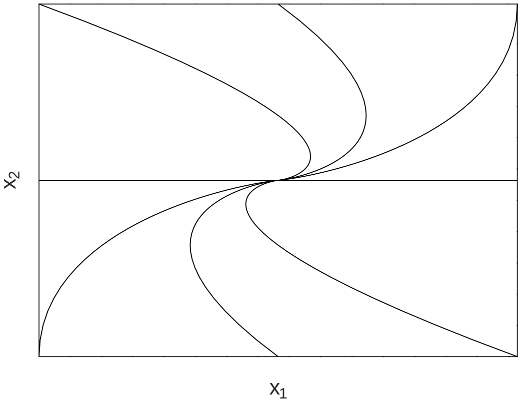
$$\begin{aligned} e^{At} &= e^{(A-\lambda I)t+\lambda It} \\ &= e^{Nt}e^{\lambda t} \\ &= (I + Nt + \frac{1}{2}N^2t^2 + \cdots)e^{\lambda t} \\ &= (I + Nt)e^{\lambda t} \\ &= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \end{aligned}$$

Degenerate eigenvalues cause polynomial changes in the growth/decay, this is particularly important when there is a zero eigenvalue.

Degenerate stable node

Take $\lambda = -1$ above, the solutions of the previous ODE are of the form

$$x_1(t) = ae^{-t} + bte^{-t} \quad \text{and} \quad x_2(t) = be^{-t}.$$



Larger Jordan blocks (real eigenvalue)

Let's look at a general $k \times k$ **Jordan block** associated with a real eigenvalue λ

$$A = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}.$$

As before $N = A - \lambda I$ is nilpotent, now with order k, $N^k = 0$.

Larger Jordan blocks (real eigenvalue)

Observe that

$$N = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \; N^2 = \begin{bmatrix} 0 & 0 & 1 & & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & & 1 \\ \vdots & & & & & 0 \\ 0 & \cdots & \cdots & & & 0 \end{bmatrix}$$

up to

$$N^{k-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \quad \text{and} \quad N^k = 0.$$

Larger Jordan blocks (real eigenvalue)

Then the matrix exponential gives

$$e^{At} = e^{\lambda t}(I + Nt + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)!})$$

or, explicitly,

$$e^{At} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{k-2}}{(k-2)!} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 1 & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Or, in other words, the solutions of the ODE $\dot{x} = Ax$ are

$$x(t) = P(t)e^{\lambda t}e_1 + P'(t)e^{\lambda t}e_2 + \cdots + P^{(k-1)}(t)e^{\lambda t}e_k$$

where P is any polynomial of order at most $k-1$.

Larger Jordan blocks (complex eigenvalue)

For a repeated complex eigenvalue can either allow for matrices with complex entries, or to keep real entries, use the complex **Jordan block** form

$$A = \begin{bmatrix} D & I_{2\times 2} & 0 & \cdots & 0 \\ 0 & D & I_{2\times 2} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & D & I_{2\times 2} \\ 0 & \cdots & \cdots & 0 & D \end{bmatrix}.$$

which is a $2\ell \times 2\ell$ matrix, $I_{2\times 2}$ are 2×2 identity matrices, 0 are 2×2 zero matrices, and D is a 2×2 matrix of the form

$$D = \begin{bmatrix} \rho & -\omega \\ \omega & \rho \end{bmatrix}.$$

Larger Jordan blocks (complex eigenvalue)

As before can separate $D = \rho I_{2\times 2} + \Omega$ with the skew symmetric 2×2 matrix $\Omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ and write

$$A = \rho I + \mathrm{diag}(\Omega, \dots, \Omega) + N$$

where the matrix N is nilpotent of order ℓ so solutions are of the form

$$x_k(t) = P_1^{(k-1)}(t) e^{\rho t} \cos \omega t + P_2^{(k-1)}(t) e^{\rho t} \sin \omega t$$

where P_1 and P_2 are polynomials of order at most $\ell - 1$.

Note that this gives a solution space of dimension 2ℓ as expected.

Commutation

Problem (Not formally assigned)

Suppose that A is a $k \times k$ matrix and B_{ij} are $k \times k$ matrices for $1 \le i, j \le m$ which each commute with A. Show that the $mk \times mk$ block matrices commute

$$\mathcal{A} = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mm} \end{bmatrix}.$$

Larger Jordan blocks (real eigenvalue)

The relevant computation is

$$N^2 = \begin{bmatrix} 0 & 0 & I_{2\times 2} & & \cdots & 0 \\ 0 & 0 & 0 & I_{2\times 2} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & & I_{2\times 2} \\ \vdots & & & & & 0 \\ 0 & \cdots & \cdots & & & 0 \end{bmatrix}$$

up to

$$N^{\ell-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & I_{2\times 2} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \quad \text{and} \quad N^{\ell} = 0.$$

Jordan normal form

Let A be an $n \times n$ real matrix with, possibly repeated, real eigenvalues $\lambda_1, \dots, \lambda_k$ and complex eigenvalues $\lambda_{k+1}, \bar{\lambda}_{k+1} \dots, \lambda_{k+\ell}, \bar{\lambda}_{k+\ell}$ with $\lambda_j = \rho_j + i\omega_j$ and $k + 2\ell = n$.

There are generalized eigenvectors $w_j = u_j + iv_j$ so that

$$Q = [u_1, \dots, u_k, u_{k+1}, v_{k+1}, \dots, u_{k+\ell}, v_{k+\ell}] \text{ is invertible}$$

and Q transforms A to block diagonal form

$$Q^{-1}AQ = \mathrm{diag}(J_1, \dots, J_m)$$

where each J_i is a **Jordan block** associated with a real or complex eigenvalue.

Stability / instability of autonomous linear systems

Stable/unstable/center subspaces

Consider the linear system

$$\dot{x} = Ax$$

with eigenvalues $\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \bar{\lambda}_{k+1} \ldots, \lambda_{k+\ell}, \bar{\lambda}_{k+\ell}$, possibly repeated, with $\lambda_j = \rho_j + i\omega_j$, and generalized eigenvectors $w_j = u_j + iv_j$ forming a basis of $\mathbb{R}^n$

$$B = \{u_1, \ldots, u_k, u_{k+1}, v_{k+1}, \ldots, u_\ell, v_\ell\}.$$

Then define the **stable subspace** E_s, **unstable subspace** E_u, and **center subspace** E_c

$$E_s = \text{span}\{u_j, v_j : \rho_j < 0\}, \ E_u = \text{span}\{u_j, v_j : \rho_j > 0\},$$

and

$$E_c = \text{span}\{u_j, v_j : \rho_j = 0\}.$$

Example: stable and center subspace

Consider the 3×3 system, which is already in its real canonical form,

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In this case eigenvalues are $\pm i$ and -1 and

$$E_s = \mathrm{span}(e_3), \;\; E_u = \{0\}, \;\; \text{and} \;\; E_c = \mathrm{span}(e_1, e_2).$$

Solutions are of the form

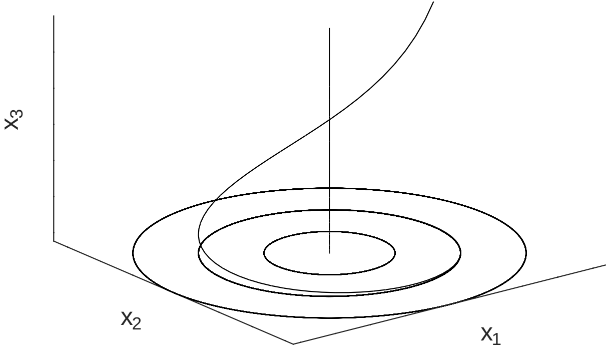
$$x(t) = [r\cos(t+\theta), r\sin(t+\theta), ze^{-t}]^T$$

parameters (r, θ, z) are the cylindrical coordinates of x_0.

Example: stable and center subspace

$$x(t) = [r\cos(t+\theta), r\sin(t+\theta), ze^{-t}]^T$$

Solutions starting above xy-plane converge to a **limit cycle** on the xy-plane as $t \to \infty$.



Invariance of stable/unstable/center subspaces

Theorem

Let A be an $n \times n$ matrix then

$$\mathbb{R}^n = E_s \oplus E_u \oplus E_c$$

and the subspaces E_s, E_u and E_c are all invariant under the flow e^{At}.

Proof.

1. The subspaces E_s, E_u and E_c are linearly independent and span $\mathbb{R}^n$ by the set up (we started with a basis of generalized eigenvectors and just split it into three disjoint subsets).

Invariance of stable/unstable/center subspaces

Proof (ctd...)

2. Next let's check that $AE \subset E$ if E is a generalized eigenspace associated with an eigenvalue λ. Note that if $v \in E$ so $(A - \lambda I)^k v = 0$ for some minimal k then

$$(A - \lambda I)^{k-1}(A - \lambda I)v = 0$$

so $(A - \lambda I)v$ is also in E, or

$$Av \in \lambda v + E = E.$$

Since E_s, E_u and E_c are direct sums of generalized eigenspaces they are also mapped to themselves by A.

Invariance of stable/unstable/center subspaces

Proof (ctd...)

3. Finally we consider the action of e^{At}. For $v \in E_s$

$$e^{At}v = \lim_{k\to\infty}(I + A + \cdots + \frac{A^k}{k!})v$$

then $(I + A + \cdots + \frac{A^k}{k!})v$ is in E_s for all k and since subspaces of finite dimensional spaces are closed the limit $e^{At}v$ is also in E_s. The same argument applies to E_c and E_u. $\square$

Stability theorem

Theorem

The following are equivalent

- *All eigenvalues of A have negative real part.*
- *For all $x_0 \neq 0$ we have $\lim_{t\to\infty} e^{At}x_0 = 0$ and $\lim_{t\to-\infty} |e^{At}x_0| = +\infty$.*
- *There are constants M, m, a and b positive so that for all $t > 0$*

$$|e^{At}x_0| \leq Me^{-at}|x_0|$$

and for all $t < 0$

$$|e^{At}x_0| \geq me^{-bt}|x_0|.$$

Stable and unstable subspaces

Theorem

For a real matrix A there are constants M, m, a and b positive so that:

- *For all $x_0 \in E_s$ and all $t > 0$*

  $$|e^{At}x_0| \leq Me^{-at}|x_0|$$

  In this case we say that the stationary solution at 0 is **asymptotically stable**.
- *For all $x_0 \in E_u$ and all $t > 0$*

  $$|e^{At}x_0| \geq me^{bt}|x_0|.$$

Stable and unstable subspaces

Theorem (Continued...)

- *For all* $x_0 \in E_c$

$$|e^{At}x_0| \leq M(1+t^k)|x_0|$$

where k *is the largest difference between the algebraic and geometric multiplicity of any eigenvalue with* 0 *real part. If* $k = 0$ *and* E_u *is trivial then all solution are bounded and we say the system is* **stable**.

- *If* $x_0 \notin E_s \oplus E_c$ *(in particular* E_u *is non-trivial) then*

$$|e^{At}x_0| \to \infty \quad \text{as} \quad t \to +\infty.$$

Non-autonomous linear systems

Non-autonomous linear systems

Consider the non-autonomous system

$$\dot{x} = A(t)x$$

for $x \in \mathbb{R}^n$ and $A(t)$ a continuously varying $n \times n$ matrix.

For each standard basis vector $e_j = (0, \ldots, 1, \ldots, 0)^T$ the vector with 1 in the jth entry and the rest zeros, define ϕ_j to be the solution of

$$\dot{\phi}_j(t) = A(t)\phi_j(t) \ \text{ with } \ \phi_j(0) = e_j.$$

These solutions form a basis for the space of all solutions, for an arbitrary initial data $v \in \mathbb{R}^n$

$$x(t) = v_1\phi_1(t) + \cdots + v_n\phi_n(t)$$

is a solution of the ODE with $x(0) = v$ (**superposition principle**).

Fundamental matrix

Define

$$\Phi(t) = [\phi_1(t), \cdots, \phi_n(t)]$$

the matrix with the basis solutions $\phi_j(t)$ as the columns. Φ is called a **fundamental matrix** / **fundamental solution** / **principal matrix solution** for the ODE.

You can check $\Phi(0) = I$ and

$$x(t) = \Phi(t)x_0 \ \text{ is the solution of the ODE with } x(0) = x_0.$$

To solve from a different initial time t_0

$$x(t) = \Phi(t)\Phi(t_0)^{-1}x_0.$$

Variation of parameters / Duhamel again

The fundamental solution Φ associated with the homogeneous system can be used also to solve inhomogeneous systems

$$\dot{x} = A(t)x + f(t).$$

The variation of parameters / Duhamel's formula in this case is

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t)\Phi(s)^{-1} f(s)\, ds.$$

Proof is a direct computation (and application of uniqueness), but let's see some motivated derivations too.

Variation of parameters derivation

In this derivativation let's make the ansatz that

$$x(t) = \Phi(t)c(t) \ \text{ for some } c \text{ with } \ c(0) = x_0,$$

which is where the name "variation of parameters" comes from. Differentiating we find

$$\dot{x}(t) = \dot{\Phi}(t)c(t) + \Phi(t)\dot{c}(t) = A(t)\Phi(t)c(t) + \Phi(t)\dot{c}(t)$$

which, in order that $x(t)$ solve the inhomogeneous ODE, must satisfy

$$A(t)\Phi(t)c(t) + \Phi(t)\dot{c}(t) = A(t)\Phi(t)c(t) + f(t)$$

or

$$\dot{c}(t) = \Phi(t)^{-1}f(t).$$

This can be integrated in t to find $c(t)$ and then $x(t)$.

Wronskian

Let $\Phi(t) = [\phi_1, \ldots, \phi_n]$ be a fundamental matrix associated with the equation $\dot{\Phi} = A(t)\Phi$. The **Wronskian** is

$$W(t) := \det(\Phi(t)).$$

Then we have **Liouville / Abel's formula**

$$\dot{W}(t) = \mathrm{tr}(A(t))W(t) \text{ and so } W(t) = W(t_0)e^{\int_{t_0}^t \mathrm{tr}(A(s))\, ds}.$$

Remark

Our previous formula for the evolution of the Jacobian determinant $\det(D\phi_t(x))$ of the flow map of a nonlinear system is a special case of Liouville / Abel's formula ($D\phi_t(x)$ was a fundamental matrix for a non-autonomous linear system).

Liouville / Abel's formula

We just apply Jacobi's formula again

$$\begin{aligned}\frac{d}{dt}W(t) &= \det{}'(\Phi(t))[\dot{\Phi}(t)]\\ &= \det(\Phi(t))\mathrm{tr}(\Phi(t)^{-1}\dot{\Phi}(t))\\ &= W(t)\mathrm{tr}(\Phi(t)^{-1}A(t)\Phi(t))\\ &= W(t)\mathrm{tr}(A(t))\end{aligned}$$

ending by the cyclic property of trace.

Application of Liouville / Abel's formula

For example the **Sturm-Liouville**-type equation

$$\ddot{u} + p(t)\dot{u} + q(t)u = 0.$$

Writing this as a first order system

$$\frac{d}{dt}\begin{bmatrix} u \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}\begin{bmatrix} u \\ \dot{u} \end{bmatrix}.$$

Since $\mathrm{tr}(A(t)) = 0$ in this case we can find

$$W(t) = u_1(t)\dot{u}_2(t) - \dot{u}_1(t)u_2(t) = e^{-\int_0^t p(s)\ ds}W(0)$$

where u_1 and u_2 are any two solution of the ODE.

Application of Liouville / Abel's formula

In particular, if $[u_1(0), \dot{u}_1(0)]$ and $[u_2(0), \dot{u}_2(0)]$ are linearly independent then

$$|W(t)| = |u_1(t)\dot{u}_2(t) - \dot{u}_1(t)u_2(t)| > 0 \ \text{ for all } \ t > 0.$$

Immediately we can derive that if $u_j(t) = 0$ at some time t then $\dot{u}_j(t) \neq 0$ (which also follows from uniqueness), and $u_2(t) \neq 0$.

Lemma

The zeros of u_1 and u_2 are interlaced.

Proof.

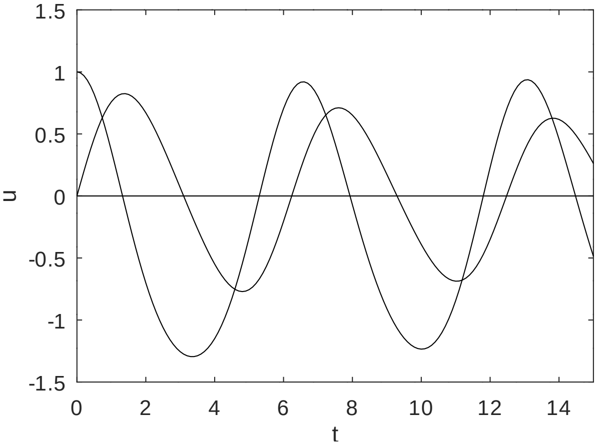
Suppose a, b are subsequent zeroes of u_1. Then $\dot{u}_1(a)$ and $\dot{u}_1(b)$ have opposite signs. Since $W(a) = \dot{u}_1(a)u_2(a) > 0$ and $W(b) = \dot{u}_1(b)u_2(b) > 0$ then $u_2(a)$ and $u_2(b)$ must also have opposite signs. That is, u_2 has a zero on (a, b). $\square$

Zero interlacement

Numerical solutions

$$\ddot{u} + \frac{1}{20}\dot{u} + (1 + \frac{1}{2}\cos(t))u = 0$$

with initial data for $[u, \dot{u}]$ chosen as $[1, 0]$ and $[0, 1]$.



Order reduction

Liouville's formula can also allow for explicit computations of solutions in certain special cases, mainly for 2×2 systems. For example if you happen to know one solution $y(t)$ of a 2×2 system

$$\dot{x} = A(t)x$$

you can pick an initial data $z(0)$ linearly independent from $y(0)$ and use Louville's formula

$$\det([y(0), z(0)])e^{\int_0^t A(s)\ ds} = y_1(t)z_2(t) - y_2(t)z_1(t)$$

solve this for (say) $z_2(t)$ in terms of $z_1(t)$ and then plug that into the ODE

$$\dot{z}_1(t) = A_{11}(t)z_1(t) + A_{12}(t)z_2(t)$$

to get a scalar ODE for z_1 which may be explicitly solvable.

Floquet Theory

Linear systems with periodic coefficients

This leads us to consider systems of the type

$$\dot{x} = A(t)x$$

with $A(t)$ a periodic function of time with minimal period $T > 0$ i.e.

$$A(t) = A(t + T) \quad \text{for all} \quad t.$$

Floquet theorem

Solutions of this equation will not generally be periodic, solutions of constant coefficient equations grow exponentially $e^{At}x_0$. Instead we can find an exponential to "mod out" by making solutions periodic.

Theorem (Floquet)

There is a matrix B and a T-periodic function $P(t)$ with $P(0) = 0$, possibly with complex entries, so that

$$\Phi(t) = P(t)e^{Bt}.$$

Furthermore there is a matrix C and a $2T$-periodic function $\tilde{P}(t)$ with $\tilde{P}(0) = 0$, both with real entries, so that

$$\Phi(t) = \tilde{P}(t)e^{Ct}.$$

1-d case

In the case of a scalar equation we can solve explicitly

$$\dot{x}(t) = a(t)x(t) \ \text{ then } \ x(t) = e^{\int_0^t a(s)\ ds} x_0.$$

Define

$$\bar{a} = \frac{1}{T}\int_0^T a(s)\ ds \ \text{ and } \ P(t) = e^{\int_0^t a(s) - \bar{a}\ ds}.$$

The function $P(t)$ is periodic with period T and

$$x(t) = P(t)e^{\bar{a}t}x_0.$$

The monodromy matrix

Let $\Phi(t)$ be a fundamental matrix for the ODE $\dot{x} = A(t)x$. Note that $\Phi(t+T)$ solves

$$\frac{d}{dt}(\Phi(t+T)) = A(t+T)\Phi(t+T) = A(t)\Phi(t+T)$$

so it is also a fundamental matrix. Define the **monodromy matrix**

$$M = \Phi(0)^{-1}\Phi(T).$$

Note that M is invertible by Liouville's formula. Since $\Phi(t)M$ is a matrix solution with the same initial data as $\Phi(t+T)$ they agree for all t

$$\Phi(t+T) = \Phi(t)M.$$

Floquet multipliers

The eigenvalues $\mu_1, \dots, \mu_n$ (possibly repeated and possibly complex) of the monodromy matrix are called the **Floquet Multipliers**. They are all nonzero because the monodromy matrix is the product of nonsingular matrices and is therefore nonsingular.

Corresponding to each eigenvalue is an eigenvector w_j (possibly complex) and defining $\chi_j(t) = \Phi(t)w_j$ we find a solution with the property

$$\chi_j(t+T) = \Phi(t+T)w_j = \Phi(t)Mw_j = \mu_j\Phi(t)w_j = \mu_j\chi_j(t).$$

Evolving over multiple periods

$$\chi_j(t+kT) = \mu_j^k\chi_j(t)$$

which converges exponentially to zero if $|\mu_j| < 1$ and converges exponentially to ∞ if $|\mu_j| > 1$.

Floquet exponents

Define the **Floquet exponents**

$$\gamma_j = \frac{1}{T}\log(\mu_j)$$

where log is the complex logarithm of $z = |z|e^{i\arg(z)}$

$$\log(z) = \log|z| + i\arg(z).$$

The argument $\arg(z)$ is the angle of z from the positive real axis, up to adding a factor $2\pi k$. Note that even when all multipliers are real, if any are negative then the corresponding exponent is complex, i.e. if $\mu_j < 0$ then

$$\log \mu_j = \log|\mu_j| + i\pi.$$

Factorization into an exponential and a periodic function

With the Floquet exponents we can write

$$\chi_j(t) = e^{\gamma_j t} p_j(t)$$

where p_j is a T-periodic (possibly complex) function. To check the periodicity

$$\begin{aligned} p_j(t+T) &= \chi_j(t+T)e^{-\gamma_j(t+T)} \\ &= \chi_j(t)\mu_j e^{-\gamma_j(t+T)} \\ &= \chi_j(t)e^{\gamma_j T} e^{-\gamma_j(t+T)} \\ &= p_j(t). \end{aligned}$$

Factorization into an exponential and a real periodic function

If all eigenvalues μ_j are real we would like to factor by a real exponential and a real periodic function, this is possible by doubling the period. We write

$$\chi_j(t) = e^{\frac{t}{T}\log|\mu_j|} q_j(t)$$

then

$$\begin{aligned} q_j(t+T) &= \chi_j(t+T) e^{-\frac{1}{T}\log|\mu_j|(t+T)} \\ &= \chi_j(t)\mu_j e^{-\frac{1}{T}\log|\mu_j|(t+T)} \\ &= \chi_j(t) e^{\log(\mu_j) - \log|\mu_j|} e^{-\frac{t}{T}\log|\mu_j|} \\ &= \mathrm{sgn}(\mu_j) q_j(t). \end{aligned}$$

Floquet theorem (back to matrices)

Now let's return to the full matrix formulation. Recall that the property of the monodromy matrix $M = \Phi(0)^{-1}\Phi(T)$

$$\Phi(t+T) = \Phi(t)M.$$

We can evolve the fundamental solution for any integer $k \geq 1$ number of periods using the monodromy matrix

$$\Phi(t+kT) = \Phi(t+(k-1)T)M = \cdots = \Phi(t)M^k.$$

This already shows an exponential type behavior over multiple periods, but we would like to see this with continuous time instead of discrete period increments.

Floquet theorem (back to matrices)

If we could find a matrix B with $e^{BT} = M$ then we would define

$$P(t) = \Phi(t)e^{-Bt}$$

and we need to check that this P is indeed T-periodic.

Floquet theorem

To check the periodicity we look at

$$
\begin{aligned}
P(t+T) &= \Phi(t+T)e^{-B(t+T)} \\
&= \Phi(t+T)e^{-B(t+T)} \\
&= \Phi(t)Me^{-BT}e^{-Bt} \\
&= \Phi(t)MM^{-1}e^{-Bt} \\
&= P(t)
\end{aligned}
$$

So we need a notion of matrix logarithm for invertible matrices to make this argument go through.

Matrix Logarithm

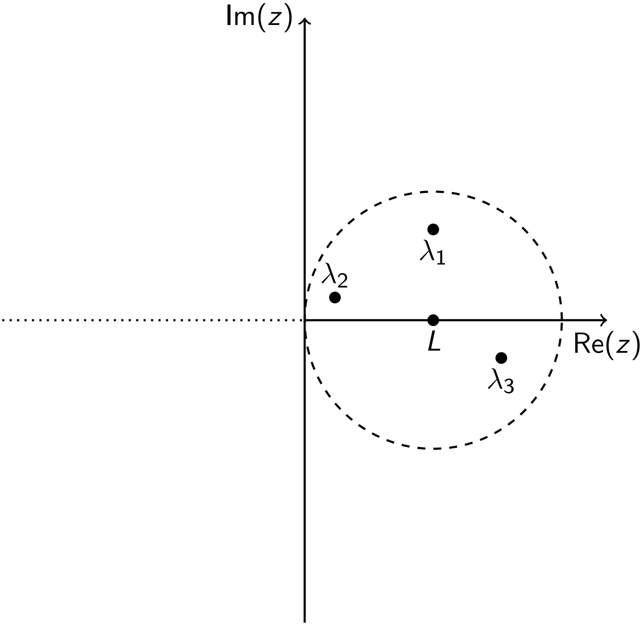
Given a matrix A we wish to find B so that

$$A = e^B.$$

This could be called the **matrix logarithm** $B = \log(A)$. Since matrix exponentials are always invertible we certainly need to assume that A is invertible.

The matrix logarithm can be defined by fixing a branch of the complex logarithm and then using its local power series expansion. If the matrix A has real entries and all eigenvalues have positive real part then there is a version of $\log(A)$ with all real entries.

Matrix Logarithm



Matrix Logarithm

More precise result:

Lemma

If the matrix A has real entries and all of its real eigenvalues have positive real part (e.g. this is true for the square of any real matrix) then there is a version of $\log(A)$ with all real entries.

First we claim that it is enough to compute the (real) logarithm of the real Jordan canonical form $J = Q^{-1}AQ$. If we can do this then define

$$\log(A) = Q \log(J) Q^{-1}$$

and we check

$$e^{\log(A)} = e^{Q \log(J) Q^{-1}} = Q e^{\log(J)} Q^{-1} = QJQ^{-1} = A.$$

So this is indeed a logarithm of A.

Matrix Logarithm

Reduction to single block

Now we are working with a matrix in real Jordan canonical form,

$$J = \mathrm{diag}(J_1, \ldots, J_m)$$

where each J_i is a Jordan block.

Suppose we can compute $\log(J_i)$ for each of the Jordan blocks. Then define

$$\log(J) = \mathrm{diag}(\log(J_1), \ldots, \log(J_m)).$$

Note that

$$\mathrm{diag}(A_1, \ldots, A_m)^k = \mathrm{diag}(A_1^k, \ldots, A_m^k)$$

for any block diagonal matrix with square blocks A_i.

Matrix Logarithm

Reduction to single block

We can check

$$
\begin{aligned}
e^{\log(J)} &= \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{diag}(\log(J_1), \dots, \log(J_m))^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \operatorname{diag}(\log(J_1)^k, \dots, \log(J_m)^k) \\
&= \operatorname{diag}(\sum_{k=0}^{\infty} \frac{1}{k!} \log(J_1)^k, \dots, \sum_{k=0}^{\infty} \frac{1}{k!} \log(J_m)^k) \\
&= \operatorname{diag}(e^{\log(J_1)}, \dots, e^{\log(J_m)}) \\
&= J.
\end{aligned}
$$

Matrix Logarithm

Single block with real eigenvalue

Consider a single $k \times k$ Jordan block with a real eigenvalue λ

$$J = \lambda I + N$$

with N nilpotent of order k.

Writing $J = \lambda(I + \frac{1}{\lambda}N)$ and inspired by the power series

$$\log(1 + x) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j} x^j \quad \text{converging for} \quad |x| < 1,$$

we define

$$\log(J) = \log(\lambda) I + \sum_{j=1}^{k-1} \frac{(-1)^j}{j \lambda^j} N^j.$$

If $\lambda > 0$ this is a real matrix. Checking $e^{\log(J)} = J$ is then a matter of power series algebra.

Matrix Logarithm

Single block with complex eigenvalue pair

Now consider a single $2k \times 2k$ real Jordan block for a complex conjugate eigenvalue pair $\rho \pm i\omega$

$$J = \mathrm{diag}(D, \dots, D) + N$$

with N nilpotent of order k and

$$D = \begin{bmatrix} \rho & -\omega \\ \omega & \rho \end{bmatrix}.$$

We rewrite $\rho \pm i\omega$ in the polar form $\rho \pm i\omega = re^{\pm i\theta}$ and then

$$D = r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Matrix Logarithm

Single block with complex eigenvalue pair

In this form we recognize that

$$\log D = \log(r) I_{2\times 2} + \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}.$$

Using the shorthand $\bar{D} = \mathrm{diag}(D, \dots, D)$ we write

$$J = \bar{D}(I + \bar{D}^{-1} N).$$

Using the commutation of $\bar{D}$ and N, the matrix $\bar{D}^{-1}N$ is still nilpotent of order k. We can define the logarithm

$$\log J = \mathrm{diag}(\log D, \dots, \log D) + \sum_{j=1}^{k-1} \frac{(-1)^j}{j} \bar{D}^{-j} N^j.$$

Floquet theorem (again)

So we have now proven the Floquet theorem:

Theorem (Floquet)

There is a matrix B and a T-periodic function $P(t)$ with $P(0) = 0$, possibly with complex entries, so that

$$\Phi(t) = P(t)e^{Bt}.$$

Furthermore there is a matrix C and a $2T$-periodic function $\tilde{P}(t)$ with $\tilde{P}(0) = 0$, both with real entries, so that

$$\Phi(t) = \tilde{P}(t)e^{Ct}.$$

Floquet stability

Since continuous periodic functions are bounded the stability of the flow map $\Phi(t) = P(t)e^{Bt}$ is determined by the constant coefficient matrix exponential e^{Bt}.

Corollary

If all Floquet exponents have negative real part (i.e. all Floquet multipliers have $|\mu_j| < 1$) then there are M and a positive so that

$$|\Phi(t)x_0| \leq Me^{-at}$$

i.e. 0 is **asymptotically stable**. *If all Floquet multipliers have $|\mu_j| \leq 1$ (i.e. all Floquet exponents have $\mathrm{Re}(\gamma_j) \leq 0$) and all multipliers with $|\mu_j| = 1$ have non-degenerate eigenspaces the the system is* **stable** *i.e.*

$$\Phi(t)x_0 \ \text{ is bounded for all } t > 0 \text{ and all } x_0.$$

A trick for showing instability

In general showing stability/instability requires integrating the ODE over one period for the fundamental solution $\Phi(T)$. For 2×2 systems sometimes we can get away with knowing only one solution by using the Wronskian / reduction of order trick. We will see that later, let's see a simple trick which will sometimes work for general $n \times n$ systems.

A trick for showing instability

Recall that, by Liouville's formula,

$$\det(M) = \frac{\det(\Phi(T))}{\det(\Phi(0))} = e^{\int_0^T \operatorname{tr}(A(s))\ ds}$$

since the determinant is the product of the eigenvalues

$$\det(M) = e^{\sum_{j=1}^n T\gamma_j} = e^{\int_0^T \operatorname{tr}(A(s))\ ds}.$$

So

$$\operatorname{Re}(\sum_{j=1}^n \gamma_j) = \frac{1}{T}\int_0^T \operatorname{tr}(A(s))\ ds.$$

Thus if $\frac{1}{T}\int_0^T \operatorname{tr}(A(s))\ ds > 0$ then there is at least one Floquet exponent with positive real part and the system is unstable for $t > 0$.

One motivation: linearizing near a limit cycle

Consider a nonlinear autonomous equation

$$\dot{x} = f(x)$$

which has a non-trivial periodic orbit x with minimal period T

$$x(t + T) = x(t) \ \text{ for all } \ t \in \mathbb{R}.$$

Consider now solutions which start at a point z_0 near the trajectory Γ_x. By making a time translation of x we can assume $|z_0 - x(0)|$ is small. We write

$$z(t) = x(t) + \varepsilon y(t)$$

and look for the equation solved by $y(t)$.

One motivation: linearizing near a limit cycle

We compute

$$\dot{y}(t) = \frac{1}{\varepsilon}(\dot{z}(t) - \dot{x}(t)) = \frac{1}{\varepsilon}\Big(f(x(t) + \varepsilon y(t)) - f(x(t))\Big).$$

Using the Taylor expansion centered at $x(t)$ on the right

$$\dot{y}(t) = Df(x(t))y(t) + O(\varepsilon|y(t)|^2).$$

If we ignore the higher order terms, which would need to be justified, we would find a linear equation with a T-periodic coefficient matrix $A(t) = Df(x(t))$, this is called the **linearization**.

Degeneracy of the linearization

Now, if we can compute the associated monodromy matrix we can classify the **linear stability** of the limit cycle $x(t)$.

One thing to note is that the time translation symmetry of the underlying periodic orbit $x(t)$ always generates a periodic solution of the linearization, and hence an eigenvalue 1 of the monodromy matrix. Precisely this is

$$z(t) = \dot{x}(t)$$

which always solves

$$\dot{z}(t) = \ddot{x}(t) = \frac{d}{dt} f(x(t)) = Df(x(t))z(t).$$

Degeneracy of the linearization

This degeneracy caused by the time translations does not rule out (linear) asymptotic stability of the periodic orbit as long as the center subspace E_c for M is only 1-dimensional and the unstable subspace E_u is trivial.

Vaguely explained reason: when $y(0) \notin E_c(0)$ may need to slightly perturb the initial time to εt_0 so that

$$\tilde{y}_0 = x(0) + \varepsilon y(0) - x(\varepsilon t_0) \in E_s(\varepsilon t_0).$$

We will really get into nonlinear stability later, this is just an idea.

Contrived example

Let's start with a simple, but contrived, example of a limit cycle. In (r, θ) polar coordinates on the (x_1, x_2) plane

$$\dot{r} = r(1 - r) \ \text{ and } \ \dot{\theta} = 1.$$

In these coordinates it is easy to see there is a 2π-periodic orbit when $r = 1$ and we can guess that it is stable based on the stability of the fixed point at 1 for the r equation. Still let's transform to (x_1, x_2) coordinates and try to apply Floquet Theory ideas.

Contrived example

Transforming to (x_1, x_2) coordinates we find

$$\begin{aligned}
\dot{x}_1 &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\
&= r(1-r)\cos\theta - r\sin\theta \\
&= (1-(x_1^2+x_2^2)^{1/2})x_1 - x_2.
\end{aligned}$$

and similarly

$$\dot{x}_2 = (1-(x_1^2+x_2^2)^{1/2})x_2 + x_1.$$

The periodic solution of this equation is

$$\xi(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}.$$

Contrived example

To compute the linearization we need $Df(x)$

$$Df(x) = \begin{bmatrix} -\frac{x_1^2}{r} + (1-r) & -1 - \frac{x_1 x_2}{r} \\ 1 - \frac{x_1 x_2}{r} & -\frac{x_2^2}{r} \end{bmatrix}$$

Evaluating along the periodic solution, which is on the set $r = 1$, we find

$$A(t) := Df(\xi(t)) = \begin{bmatrix} -\cos^2(t) & -1 - \cos(t)\sin(t) \\ 1 - \cos(t)\sin(t) & -\sin^2(t) \end{bmatrix}$$

So the linearization around this periodic solution is the ODE

$$\dot{y}(t) = A(t)y(t).$$

Contrived example

We know there is always one eigenvalue 1 of the monodromy matrix and a periodic solution of the linearization corresponding to the time translation symmetry

$$z(t) = \frac{d}{dt}\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}.$$

If $y(t)$ is the solution with initial data $[0,1]^T$ then $\Phi(t) = [z(t), y(t)]$ is a fundamental solution and $M = \Phi(2\pi)$ is a monodromy matrix. To find the other eigenvalue of M we write out the characteristic polynomial

$$0 = \lambda^2 - \mathrm{tr}(M)\lambda + \det(M).$$

Contrived example

We can compute the determinant of M using Liouville's formula

$$\begin{aligned} \det(M) &= \det(\Phi(t)) \\ &= \det(\Phi(0)) e^{\int_0^t \operatorname{tr} A(s) ds} \\ &= e^{-2\pi} \end{aligned}$$

Since we also know that $\lambda = 1$ is one of the two roots of $\det(M - \lambda I)$ is one we can solve for the trace

$$\operatorname{tr}(M) = 1 + e^{-2\pi}$$

and the other root

$$\lambda = e^{-2\pi}.$$

The corresponding Floquet exponent is

$$\gamma = \frac{1}{2\pi} \log(\lambda) = -1.$$

Hill's equation

A general form of equation for which Floquet theory is useful is **Hill's equation**

$$\ddot{x} + q(t)x = 0 \ \text{ with } q \text{ a } T\text{-periodic function.}$$

Originated by G.W. Hill in 1877 in a study of (linearized) stability of the lunar orbit, attempting to explain motion of the lunar perigee. Many other specific equations which fall under this form as well (e.g. **Mathieu equation**). As a 2×2 system becomes

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

Hill's equation

If Φ is the principal fundamental solution and $M = \Phi(T)$ is the monodromy matrix then, by Liouville's formula,

$$\det(M) = \det(\Phi(0)) = 1.$$

Thus the characteristic equation of M is, in terms of $\Delta = \frac{1}{2}\mathrm{tr}(M)$,

$$\mu^2 - 2\Delta\mu + 1 = 0$$

and the roots are

$$\mu_\pm = \Delta \pm \sqrt{\Delta^2 - 1}.$$

Hill's equation

Case 1: $(\Delta^2 > 1)$ In this case both

$$\mu_\pm = \Delta \pm \sqrt{\Delta^2 - 1}$$

are real and have the same sign $\sigma = \mathrm{sgn}(\Delta)$.

Since $\mu_+\mu_- = 1$ one multiplier is larger and one is smaller than 1 in absolute value.

There are two linearly independent solutions of the form

$$\chi_\pm(t) = e^{\frac{1}{T}\log|\mu_\pm| t} p_\pm(t) \;\text{ with }\; p_\pm(t+T) = \sigma p(t).$$

Thus the system is **unstable** in this case.

Hill's equation

Case 2: $(\Delta^2 < 1)$ In this case there are two complex eigenvalues

$$\mu_\pm = \Delta \pm i\sqrt{1-\Delta^2}$$

with real part $\Delta < 1$. Both eigenvalues have complex modulus $|\mu_\pm| = 1$.

There are two linearly independent solutions of the form

$$\chi_\pm(t) = e^{\frac{1}{T} i \arg(\mu_\pm) t} p_\pm(t) \text{ with } p_\pm(t+T) = p(t)$$

where arg is a branch of the complex argument.

Thus the system is stable (but not asymptotically stable) in this case.

Hill's equation

Case 3: $(\Delta^2 = 1)$ In this case there is a single real eigenvalue

$$\mu = \Delta$$

and there are two solutions $p_\pm$ with

$$p_\pm(t+T) = \sigma p_\pm(t)$$

either T periodic or $2T$ periodic depending on the sign of Δ. Or, if the multiplier is degenerate the two solutions have the form

$$p_+(t) \;\text{ and }\; p_+(t) + tp_-(t).$$

Thus the stability of the system depends on the degeneracy of the multiplier.

Hill's equation stability

Theorem

Hill's equation is stable if $|\Delta| < 1$ *and unstable if* $|\Delta| > 1$*.*

Matheiu's equation

Matheiu's equation

$$\ddot{x} + \omega^2(1 + \varepsilon \cos(t))x = 0$$

arises in modeling the motion of a charged particle moving in the electric field of a quadropole with an oscillating voltage.

Since stability depends strongly on ω and ε, and ω depends on the particle mass this system can be used to filter ions by mass (technically mass/charge ratio) .

Matheiu's equation

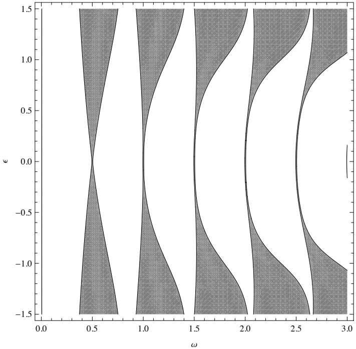


Figure: Stability diagram for Matheiu's equation, shaded regions $|\Delta(\omega, \varepsilon)| > 1$ are unstable. Note that these branch from $\omega \in \frac{1}{2}\mathbb{N}$ where $|\Delta(\omega, 0)| = |\cos(2\pi\omega)| = 1$.

Teschl, *Ordinary Differential Equations and Dynamical Systems*, page 96.

Periodic orbits of forced linear systems

Consider a time-periodic forced system

$$\dot{x} = A(t)x + b(t) \tag{1}$$

where both A and b have period T (not necessarily the minimal period for either).

Does this system have a T-periodic solution?

Theorem

If 1 is not a Floquet multiplier for the homogeneous system then (1) has a T-periodic solution.

Periodic orbits of forced linear systems

1. It is enough to find a solution with $x(T) = x(0)$. If we had such a solution then $x(t)$ and $x(T+t)$ are both solutions of (1) with the same initial data so, by uniqueness, they are the same.

2. We use Duhamel's formula

$$x(T) = \Phi(T)x(0) + \int_0^T \Phi(T)\Phi(s)^{-1}b(s)\, ds.$$

In order for $x(T) = x(0)$ we would need then

$$(I - \Phi(T))x(0) = \int_0^T \Phi(T)\Phi(s)^{-1}b(s)\, ds$$

and plugging back in we see that if $x(0)$ satisfies this equation then $x(T) = x(0)$.

Periodic orbits of forced linear systems

The equation

$$(I - \Phi(T))x(0) = \int_0^T \Phi(T)\Phi(s)^{-1}b(s)\ ds$$

is guaranteed to have a solution if $I - \Phi(T)$ is invertible. This is the case as long as

$$v - \Phi(T)v \neq 0 \ \text{ for } \ v \neq 0$$

i.e. $\Phi(T)$ does not have 1 as an eigenvalue.

$\square$

Nonlinear stability of fixed points

Nonlinear stability

Consider a nonlinear autonomous system

$$\dot{x} = f(x)$$

with f smooth and $f(0) = 0$.

If all of the eigenvalues of $Df(0)$ have strictly negative real part then the fixed point at zero is said to be **linearly stable**.

In this setting we can also prove a type of nonlinear stability.

Nonlinear stability

Theorem

Suppose that f is smooth and $f(0) = 0$ and all eigenvalues of $Df(0)$ have negative real part. Then there is a $\delta > 0$ and $\alpha > 0$ so that any solution of

$$\dot{x} = f(x) \quad \text{and} \quad x(0) = x_0$$

with $|x_0| \leq \delta$ will satisfy

$$|x(t)| \leq Ce^{-\alpha t}|x_0|.$$

Perturbed linear systems

To attack this theorem will first study nonlinear perturbations of linear systems

$$\dot{x} = A(t)x + g(t,x) \quad \text{with} \quad x(0) = x_0.$$

We will be able to consider either stable constant coefficient systems or stable time periodic systems.

We encapsulate both cases with the following generalized assumption on the fundamental matrix solutions of the homogeneous equation:

Call $\Phi(t)$ to be the principal matrix solution of the homogeneous equation and $\Pi(t,s) = \Phi(t)\Phi(s)^{-1}$ to be the principal matrix solution started at time s. Assume that, for some $C \geq 2$,

$$\|\Pi(t,s)\|_{op} \leq Ce^{-\alpha(t-s)} \quad \text{for} \quad t \geq s \geq 0.$$

Perturbed linear systems

Next we consider the assumption on the perturbation term $g(t,x)$ in

$$\dot{x} = A(t)x + g(t,x) \ \text{ with } \ x(0) = x_0.$$

We assume that

$$|g(t,x)| \leq b_0|x| \ \text{ for all } \ |x| \leq \delta.$$

With these two assumptions we can prove that, if $|x_0| \leq \delta/C$ and $b_0 C < \alpha$ then

$$|x(t)| \leq De^{-(\alpha - b_0 C)t}|x_0| \ \text{ for } \ t \geq 0.$$

Perturbed linear systems
Proof

Treating the term g as a perturbation the natural thing is to use Duhamel

$$x(t) = \Pi(t,0)x_0 + \int_0^t \Pi(t,s)g(s,x(s))\,ds.$$

Let $T = \inf\{t > 0 : |x(t)| \geq \delta\} > 0$ and estimate by triangle inequality for $0 \leq t \leq T$

$$\begin{aligned} |x(t)| &\leq \|\Pi(t,0)\|_{op}|x_0| + \int_0^t \|\Pi(t,s)\|_{op}|g(s,x(s))|\,ds \\ &\leq Ce^{-\alpha t}|x_0| + \int_0^t Ce^{-\alpha(t-s)}b_0|x(s)|\,ds. \end{aligned}$$

Multiplying both sides by $e^{\alpha t}$ we see an opportunity to apply Grönwall to $z(t) = e^{\alpha t}|x(t)|$.

Perturbed linear systems

Proof

Writing in terms of $z(t) = e^{\alpha t}|x(t)|$

$$z(t) \leq C|x_0| + \int_0^t Cb_0 z(s)\, ds$$

and then applying Grönwall inequality

$$z(t) \leq C|x_0|e^{Cb_0 t}$$

or

$$|x(t)| \leq C|x_0|e^{-(\alpha - Cb_0)t}.$$

In particular if $|x_0| \leq \delta/C$ then $|x(t)| \leq \delta$ for all $t > 0$ and $T = +\infty$.

$\square$

Application to nonlinear systems

In the setting of a nonlinear system f with a linearly stable fixed point at the origin, by Taylor's theorem with remainder,

$$f(x) = Df(0)x + R(x)|x|$$

where the remainder term has the bound

$$|R(x)| \to 0 \ \text{ as } \ |x| \to 0.$$

Application to nonlinear systems

By the linear stability there are $C \geq 2$ and $\alpha > 0$ so that

$$\|e^{Df(0)t}\|_{op} \leq Ce^{-\alpha t} \quad \text{for} \quad t \geq 0.$$

Let $\delta > 0$ sufficiently small so that $|x| \leq \delta$ implies

$$|R(x)| \leq \frac{1}{2}C^{-1}\alpha.$$

Then the system

$$\dot{x} = f(x) = Df(0)x + R(x)|x|$$

fits the assumptions of the linear perturbation theorem and

$$|x(t)| \leq C|x_0|e^{-\frac{\alpha}{2}t} \quad \text{for } t > 0 \text{ as long as } \quad |x_0| \leq \delta.$$

Nonlinear stability of periodic orbits

We need some more concepts and will come back to this topic later!