Math 5750/6680-3 — Game Theory

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http://www.math.utah.edu/~ethier/5750.html

http://www.math.ucla.edu/~tom/Game_Theory/Contents.html

I. Impartial Combinatorial Games

- Two players, perfect information, win-or-lose outcomes, no chance moves (**combinatorial**).
- Both players have the same options from each position (**impartial**).
- Players alternate moves. Under **normal play rule** the last player to move wins. Under **misère play rule** the last player to move loses.

Example

Chomp. Start with an $m \times n$ board, say 4×7 to be specific. A legal move consists of removing any square and all squares to the right and above. The player who removes the lower left square loses.

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II. Two-Person Zero-Sum Games

- Also known as *matrix games*. There are two players, Player I (row player) has m pure strategies and Player II (column player) has n pure strategies.
- The payoff matrix is
$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ m \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}.$$
- Both players choose their strategies independently and simultaneously. If Player I chooses strategy i and Player II chooses strategy j, then Player I wins a_{ij} from Player II.

II. Two-Person Zero-Sum Games, cont.

Example

Rock, Paper, Scissors. Both players have three pure strategies: rock, paper, and scissors. Rock beats scissors, scissors beats paper, and paper beats rock. So the payoff matrix is

$$\begin{array}{cc} & \begin{array}{ccc} R & P & S \end{array} \\ \begin{array}{c} R \\ P \\ S \end{array} & \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}.$$

The optimal strategy, for both players, is the mixed (or randomized) strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

See `http://www.worldrps.com`

III. Two-Person General-Sum Games

- Also known as *bimatrix games*. Same as matrix games except the payoff matrix is replaced by a payoff bimatrix:

$$\begin{array}{c c}
 & \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \cdots & \quad n \quad \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ m \end{array} & \begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & \cdots & (a_{1n}, b_{1n}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & \cdots & (a_{2n}, b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{m1}, b_{m1}) & (a_{m2}, b_{m2}) & \cdots & (a_{mn}, b_{mn}) \end{pmatrix}.
\end{array}$$

- If Player I chooses strategy i and Player II chooses strategy j, then Player I wins a_{ij} and Player II wins b_{ij}.

III. Two-Person General-Sum Games, cont.

Example

The Prisoners' Dilemma. Two well-known crooks are captured and separated into different rooms. The district attorney knows he does not have enough evidence to convict on the serious charge of his choice, but offers each prisoner a deal. If just one of them will turn state's evidence (i.e., rat on his confederate), then the one who confesses will be set free, and the other sent to jail for the maximum sentence. If both confess, they are both sent to jail for the minimum sentence. If both exercise their right to remain silent, then the district attorney can still convict them both on a very minor charge.

$$\begin{array}{r c} & \begin{array}{cc} \text{silent} & \text{confess} \end{array} \\ \begin{array}{r} \text{silent} \\ \text{confess} \end{array} & \begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix} \end{array}$$

Both prisoners confessing is a Nash equilibrium, but not the best result.

IV. Cooperative Games in Coalitional Form

- The are n players, $1, 2, \ldots, n$. Let $N = \{1, 2, \ldots, n\}$ be the *grand coalition*.
- A subset $S \subset N$ is called a *coalition*. Associated with each coalition is its *worth* $v(S)$. The function v from 2^N into $\mathbf{R}$ is called the *characteristic function*.
- A *simple* game is one for which $v(S) = 0$ or 1 for every S. If $v(S) = 1$, S is called a *winning coalition*. If $v(S) = 0$, S is called a *losing coalition*.

Example

- The UN Security Council has 5 permanent members (China, France, Russian Federation, UK, and US) and 10 nonpermanent members.
- To pass a resolution, 9 votes are needed, including those of all 5 permanent members. (Each permanent member has veto power.)
- Equivalent description: Each permanent member has 7 votes, and each nonpermanent member has 1 vote. 39 votes are needed for passage of a resolution.
- How much voting power does each member have?
- The Shapley–Shubik power index gives one possible answer. Each permanent member has voting power 19.6 percent, and each nonpermanent member has voting power 0.2 percent.

`http://www.un.org/en/sc/`