Math 5270
Transformational Geometry

Day 8

Summer 13

Fill out your table

There seem to be two types of maps present. From your work, try to distinguish the two categories. Can you write a ~~generilized~~ generalized statement and prove any of the claims you make?

Maps in the first category, ones with the property that

$$f\big((a,b)+(c,d)\big) = f(a,b) + f(c,d)$$

$$f\big(t(a,b)\big) = t f(a,b)$$

are called LINEAR TRANSFORMATIONS.

11 Monday

1. By any means necessary fill out the following table:

$(2,0)+(0,3)=(2,3)$ $\quad$ $-3\cdot(2,0)=(-6,0)$

	f	$f(2,0)$	$f(0,3)$	$f(2,3)$	$f(-6,0)$
✓	$(x,y)\mapsto(\frac{3}{5}x-\frac{4}{5}y,\frac{4}{5}x+\frac{3}{5}y)$	$(\frac{6}{5},\frac{8}{5})$ +	$(-\frac{12}{5},\frac{9}{5})$ =	$(-\frac{6}{5},\frac{17}{5})$	$(-\frac{18}{5},\frac{24}{5}) = (-3)\cdot(\frac{6}{5},\frac{8}{5})$
✓	$(x,y)\mapsto(-\frac{3}{5}x+\frac{4}{5}y,\frac{4}{5}x+\frac{3}{5}y)$	$(-\frac{6}{5},\frac{8}{5})$ +	$(\frac{12}{5},\frac{9}{5})$ =	$(\frac{6}{5},\frac{17}{5})$	$(\frac{18}{5},-\frac{24}{5})$
✓	$(x,y)\mapsto(-2x,\frac{1}{2}y)$	$(-4,0)$ +	$(0,\frac{3}{2})$ =	$(-4,\frac{3}{2})$	$(12,0)$
✓	$(x,y)\mapsto(y,x)$	$(0,2)$ +	$(3,0)$ =	$(3,2)$	$(0,-6)$
✗	translation $(x,y)\mapsto(x+2,y+1)$	$(4,1)$	$(2,4)$	$(4,4)$	$(-4,1)$
✓	$(x,y)\mapsto(-x,y)$	$(-2,0)$ +	$(0,3)$ =	$(-2,3)$	$(6,0)$
✓	$(x,y)\mapsto(2x,2y)$	$(4,0)$ +	$(0,6)$ =	$(4,6)$	$(-12,0)$
✗	some kind of collapse onto a line $(x,y)\mapsto(x-y+1,-x+y-2)$	$(3,-4)$	$(-2,1)$	$(0,-1)$	$(-5,4)$

There seem to be two types of maps present. From your work, try to distinguish the two categories. Can you write a generilized statement and prove any of the claims you make?

✓:
$$f\big((a,0)+(0,b)\big)=f(a,0)+f(0,b)$$
$$f\big(t(a,0)\big)=tf(a,0)$$

Modified conjecture:

Let's check points of the form (a,b) & (c,d) for 3^{rd} map:

$$f\big((a,b)+(c,d)\big)=f(a,b)+f(c,d)$$
and
$$f\big(t(a,b)\big)=tf(a,b)$$

For translation:

$f((a,b)) + f((c,d)) \neq f((a,b)+(c,d)) = f((a+c, b+d))$

+ const factor — const. factor — const. factor

$(a+2, b+1) + (c+2, d+1) \neq (a+c+2, b+d+1)$

2. Vectors and linear combinations

   1. Draw the following segments. What do they have in common? from $(3,-1)$ to $(10,3)$; from $(1.3,0.8)$ to $(8.3,4.8)$; from $(\pi,\sqrt{2})$ to $(7+\pi, 4+\sqrt{2})$

      (a) Find another example of a directed segment that represents this vector. The initial point of your segment is called the tail of the vector, and the final point is called the head.

      (b) Which of the following directed segments represents $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$? from $(-2,-3)$ to $(5,-1)$; from $(-3.-2)$ to $(11,6)$; from $(10,5)$ to $(3,1)$; from $(-7,-4)$ to $(0,0)$?

      (c) Brief discussion

   2. Given the vector $\begin{bmatrix} -5 \\ 12 \end{bmatrix}$, find the following vectors:

      (a) same direction, twice as long

      (b) same direction, length 1

      (c) opposite direction, length 10

      (d) opposite direction, length c

   3. Addition of vectors

   4. Real vector spaces - definition

3. Directions

   - Through the origin
   - Linear independence
   - Generalized directions
   - Parallels

4. Vector Thales

5. Prove: Diagonals of parallelogram bisect each other

6. Centroid of a triangle

I can write this:

$$f(\underbrace{t(a,b)+s(c,d)}_{\text{linear combination of two points}}) = tf(a,b) + sf(c,d)$$

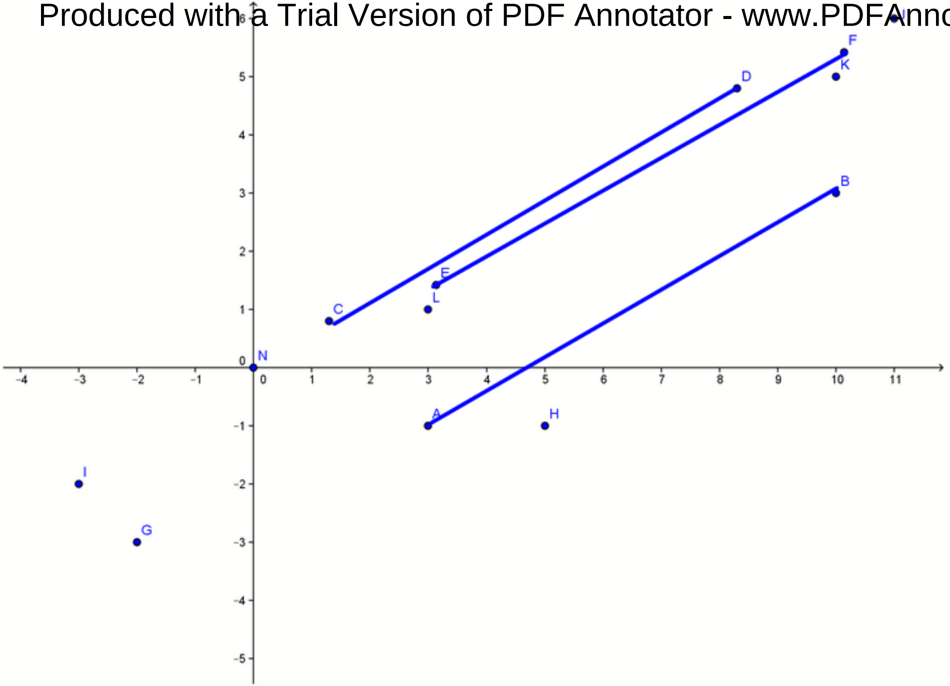
$$f(t(a,b)) + f(s(c,d))$$

b/c of second eq.

linear transformation

Commonalities

Draw the following segments. What do they have in common?
from $(3, -1)$ to $(10, 3)$; from $(1.3, 0.8)$ to $(8.3, 4.8)$; from $(\pi, \sqrt{2})$ to $(7 + \pi, 4 + \sqrt{2})$


A
B
C
D
E
F
G
H
I
K
L
N
-4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11
-5 -4 -3 -2 -1 0 1 2 3 4 5 6

Commonalities

+7 +7

Draw the following segments. What do they have in common? from $(3, -1)$ to $(10, 3)$; from $(1.3, 0.8)$ to $(8.3, 4.8)$; from $(\pi, \sqrt{2})$ to $(7 + \pi, 4 + \sqrt{2})$

+4 +4

The directed segments represent the vector $[7, 4]$, also denoted by $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ The components of the vector are 7 and 4.

$\begin{bmatrix} 7 \\ 4 \end{bmatrix}$

(0,0) to (7,4) ; (1,1) to (8,5) ; (a,b) to (a+7, b+4)

(a) Find another example of a directed segment that represents this vector. The initial point of your segment is called the tail of the vector, and the final point is called the head.

(b) Which of the following directed segments represents $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$? from $(-2,-3)$ to $(5,-1)$; from $(-3.-2)$ to $(11,6)$; from $(10,5)$ to $(3,1)$; from $\boxed{(-7,-4) \text{ to } (0,0)}$?

-4   -7

(c) Brief discussion

$\begin{bmatrix} -7 \\ -4 \end{bmatrix}$

Scalar multiples

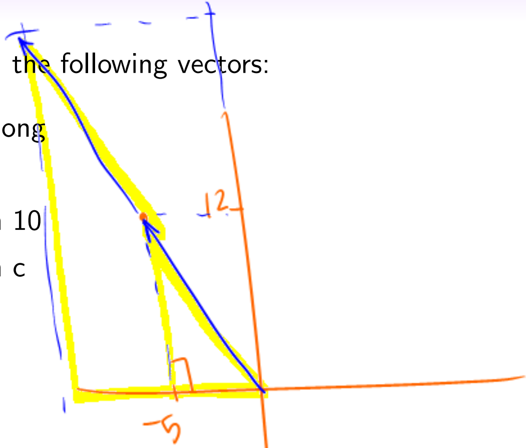
Given the vector $\begin{bmatrix} -5 \\ 12 \end{bmatrix}$, find the following vectors:

(a) same direction, twice as long

(b) same direction, length 1

(c) opposite direction, length 10

(d) opposite direction, length c



a) $2\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -10 \\ 24 \end{bmatrix}$

b) $\frac{1}{13}\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{5}{13} \\ \frac{12}{13} \end{bmatrix}$

c) $\frac{-10}{13}\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{50}{13} \\ -\frac{120}{13} \end{bmatrix}$

d) $-\frac{c}{13}\begin{bmatrix} -5 \\ 12 \end{bmatrix}$

Terminology

When the components of the vector

$$\begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

are multiplied by a given number t, the result may be written either as

$$\begin{bmatrix} -5t \\ 12t \end{bmatrix} \quad \text{or as} \quad t\begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

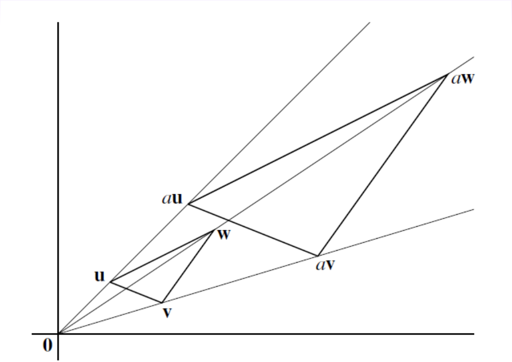
This is called the *scalar multiple* of vector $\begin{bmatrix} -5 \\ 12 \end{bmatrix}$ by the scalar t.

More generally

If $t \in \mathbf{R}$ and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, we define:

$$t\mathbf{u} = t \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} tu_1 \\ tu_2 \end{bmatrix}$$

Properties of scalar multiplication



$\mathbf{u} = \vec{u} = \overrightarrow{OP}$

For $\mathbf{u}, \mathbf{v}$ vectors and a, b real numbers we have:

$1\mathbf{u} = \mathbf{u}$

$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

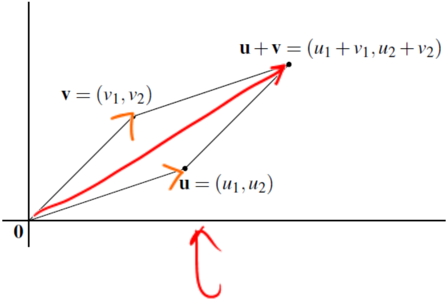
$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

$a(b\mathbf{u}) = (ab)\mathbf{u}$.

Addition of vectors

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are two vectors we define $\mathbf{u} + \mathbf{v}$:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$



Properties of addition

For $\mathbf{u}, \mathbf{v}$ vectors we have:

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

$\mathbf{u} + \mathbf{0} = \mathbf{u}$

$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Real vector space

A real vector space is a set V whose elements we'll call vectors, with operations of vector addition and scalar multiplication satisfying the following conditions:

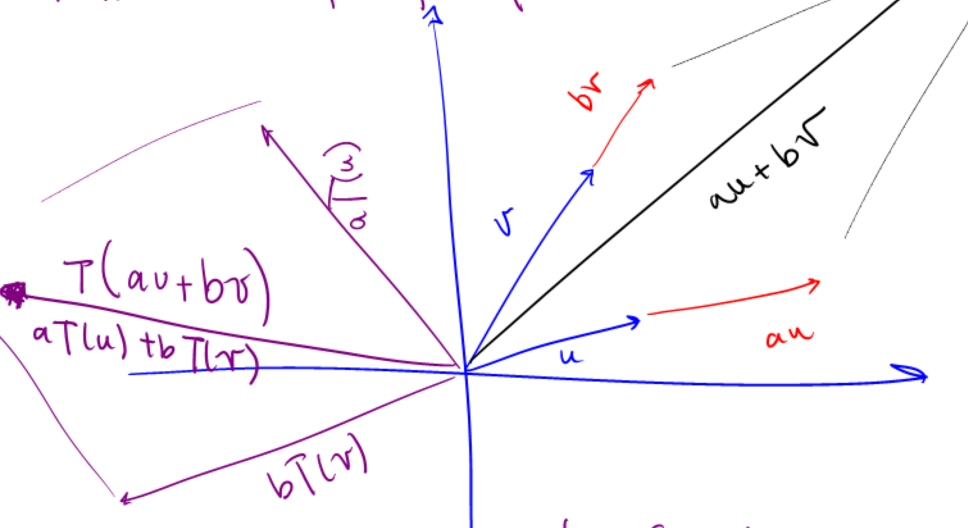
- If $\mathbf{u}$ and $\mathbf{v}$ are in V, then so are $\mathbf{u}+\mathbf{v}$ and $a\mathbf{u}$ for any real number a.
- There is a zero vector $\mathbf{0}$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for each vector $\mathbf{u}$. Each $\mathbf{u}$ in V has a additive inverse $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
- Vector addition and scalar multiplication on V have the properties:
  $1\mathbf{u}=\mathbf{u}$
  $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$
  $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$
  $a(b\mathbf{u})=(ab)\mathbf{u}$.
  $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
  $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

a transformation $T: V \to V$ is a linear transformation if for every $u, v \in V$ and $a, b \in \mathbb{R}$ we have

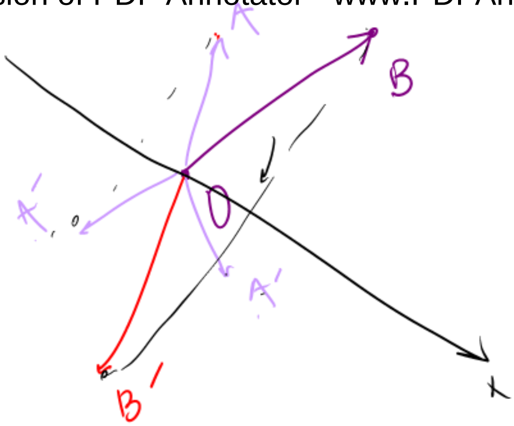
$$T(au + bv) = aT(u) + bT(v).$$

$au + bv$ is called linear combination of vectors u and v.

T is a linear transformation.



In order to define a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ what is the minimal set of information needed?



one vector isn't enough.

Two vectors:

- $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$: $\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$

Every vector, $\begin{bmatrix}x\\y\end{bmatrix}$, is a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

why is this enough?

T is linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) =$$

$$= xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

— any two nonzero vectors would work IF they're not scalar multiples of each other.

Let $\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{bmatrix} c \\ d \end{bmatrix}$ be 2 vectors which are not scalar multiples

$$\begin{bmatrix} a \\ b \end{bmatrix} m + \begin{bmatrix} c \\ d \end{bmatrix} n = \begin{bmatrix} x \\ y \end{bmatrix}$$

$n = \frac{xb - ay}{bc - ad}$

$am + cn = x$

$bm + dn = y$

$\boxed{m = \frac{dx - cy}{da - bc}}$

$da - bc \neq 0$

$da \neq bc$

$\frac{a}{c} \neq \frac{b}{d}$

$\nexists k$ st $a = ck$, $b = dk$

$\boxed{\text{so if } da \neq bc}$

we have a unique solution to the system when $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ were not scalar multiples of each other.

This is what we know:

T linear transformation

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}a\\b\end{bmatrix}, \quad a,b,c,d\in\mathbb{R}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}c\\d\end{bmatrix}$$

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=x\underbrace{T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)}+y\underbrace{T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)}=$$

$$=x\begin{bmatrix}a\\b\end{bmatrix}+y\begin{bmatrix}c\\d\end{bmatrix}=\begin{bmatrix}ax+cy\\bx+dy\end{bmatrix}$$

$$=\begin{bmatrix}a & c\\b & d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

If T_1 is a rotation about O by θ_1

T_2 is a rotation about O by θ_2,

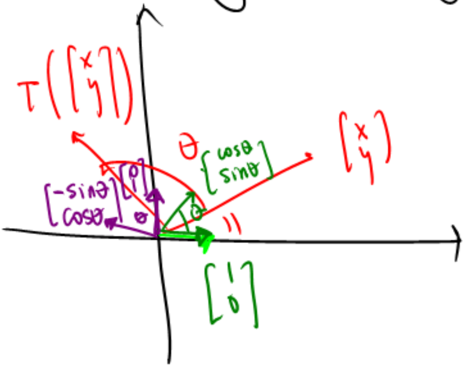
what is the matrix for

$$(T_1 \circ T_2)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Since $T_1 \circ T_2$ is rot by $\theta_1 + \theta_2$ around O

$$\begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

1) Find the matrix for $r_{0,\theta}$ (rotation about the origin by angle θ counterclockwise)



$$\begin{bmatrix}1\\0\end{bmatrix} \longmapsto \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \longmapsto \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$A\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{bmatrix} = \begin{bmatrix}cx - sy\\ sx + cy\end{bmatrix}$$

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$ matrix that corresponds to the linear transformation T.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ b \end{bmatrix} + y\begin{bmatrix} c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} ax+cy \\ bx+dy \end{bmatrix}$$

same as the "usual" multiplication of matrices

$$\boxed{T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}}$$