FRG MINI-WORKSHOP – FALL 2014
FOLIATIONS WITH EFFECTIVE ANTI-CANONICAL BUNDLE

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1. PRESENTATION

The goal of this workshop is to discuss some recent results on the classification/structure of foliations on projective varieties having effective anti-canonical bundle. Two main classes of foliations are going to be discussed.

1.1. **Fano foliations.** These are the foliations with ample anti-canonical bundle. They have been investigated by Araujo and Druel in a series of papers ([2], [4], [1], [3]).

1.2. **Foliations with trivial canonical class.** These are the foliations with numerically trivial canonical bundle. Most of the results so far have focused on codimension one foliations. The smooth case has been investigated by Touzet [29] and the general case by Loray, Pereira and Touzet [21].

2. PREPARATORY TALKS

We suggest the following topics for the preparatory talks.

2.1. **Classification of low degree foliations on projective spaces (one talk).** The idea would be to cover the classification of foliations of low degree on projective spaces obtained by Jouanolou and Cerveau-Lins Neto.

2.2. **Birational classification of foliations in dimension two (two talks).** To give some context it would be nice to review the classification of foliations in dimension two following [11].

2.3. **Rational curves on foliated varieties (one talk).** One of common starting points of the classification of Fano foliations and of foliations with trivial canonical class is Bogomolov-McQuillan Theorem.

3. SHORT GUIDE TO THE LITERATURE

3.1. **Classification of low degree foliations on projective spaces.** The study of low degree foliations on the projective plane can be traced back to Darboux [15]. Apparently inspired by Darboux's work, Jouanolou [19] started the study of codimension one foliations on projective spaces of dimension at least three. He obtained a description of the irreducible components of the space of codimension one foliations of degree zero and one.

The description of the irreducible components of the space of codimension one foliations of degree two is due to Cerveau and Lins Neto [14].

Until very recently, no attention has been paid to foliations of codimension higher than one. The description of degree zero foliations of arbitrary codimension is due to Cerveau

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and Déserti [17]. The case of degree one foliations is due to Loray, Pereira and Touzet [18] who based their approach on a classification of germs of integrable q-forms by de Medeiros [16].

3.2. **Birational classification of foliations in dimension two.** The classification of smooth foliations on compact complex surfaces (algebraic or not) is due to Brunella [8].

The first indications that a birational classification of foliations were possible appeared in L. G. Mendes Phd thesis [24]. Most of the results there presented, were subsequently refined and published in [23].

The existence of minimal models for foliations was first discussed in [9] and the foliations which do not admit a minimal model are completely classified.

Around 2000 a clear picture of what of the birational classification of foliated surfaces should be is traced by McQuillan in the IHES Preprint "Non commutative Mori theory" and further explained in Brunella's notes [10] for a course taught at the first CLAM (Congreso LatinoAmericano de Matematicos) which took place at IMPA. At that point a complete classification of foliations of non-negative Kodaira dimension was available and the only elusive case was the foliations with pseudoeffective canonical bundle but with negative Kodaira dimension (non-abundant foliations). Using Brunella's deep result on subharmonicity of the variation of the Poincaré metric [12], the elusive case was finally understood in a joint effort (but published independently) of Brunella [12] and McQuillan [22].

Perhaps the most accessible exposition of the birational classification of foliations on surfaces is Brunella's survey [11].

3.3. **Rational curves on foliated varieties.** Before Brunella, McQuillan, Mendes started to worry with the birational geometry of foliated surfaces, Miayoka in [25] (see also Shepherd–Barron's exposition on [28]) proved a semi-negativity result for the tangent sheaf of foliations in non-uniruled varieties.

Miyaoka's result was made much more precise by a result found by Bogomolov and McQuillan in [6]. Essentialy the result says that if the restriction of the tangent sheaf of a foliation to a curve is ample then every leaf of the foliation intersecting the given curve is algebraic and the general leaf intersecting the curve is rationally connected. The algebraicity of the leaves was established independently by Bost in [7]. An alternative argument for the rational connectedness of the leaves is proposed by Kebekus-Sola-Toma in [27].

3.4. **Foliations with positive anticanonical bundle.** It seems fair to say that the study of foliations with positive anticanonical bundle started with J. Wahl in [30] who proved that one dimensional foliations with ample tangent sheaf can only exist in projective spaces. Beauville proposed a generalization of Wahl's Theorem in the form of a conjecture in [5]. This conjecture was proved in [13], giving as a corollary bounds for the index of Fano foliations.

The classification of Fano foliations of high index has been pursued by Araujo–Druel in the series of papers [2], [4], [1], [3].

3.5. **Foliations with numerically trivial canonical bundle.** The study of foliations with trivial tangent sheaf on non-uniruled manifolds can be traced back to [20] where the automorphism group of these manifolds are analyzed. Smooth codimension one foliations with numerically trivial canonical bundle have been classified by Touzet in [29]. The structure of codimension one singular foliations with numerically trivial canonical bundle is the main object of study of [21].

For smooth foliations with numerically trivial canonical bundle of arbitrary codimension under the additional assumption that all the Chern classes of the tangent bundle vanish a structure theorem is provided in [26].

REFERENCES

[1] C. Araujo and S. Druel. On codimension 1 del Pezzo foliations on varieties with mild singularities. 2012.

[2] C. Araujo and S. Druel. On Fano foliations. *Adv. Math.*, 238:70–118, 2013.

[3] C. Araujo and S. Druel. Codimension 1 Mukai foliations on complex projective manifolds. 2014.

[4] C. Araujo and S. Druel. On Fano foliations 2. 2014.

[5] A. Beauville. Symplectic singularities. *Invent. Math.*, 139(3):541–549, 2000.

[6] F.A. Bogomolov and M.McQuillan. Rational curves on foliated varieties. *IHES Preprint*, 2001.

[7] J.-B. Bost. Algebraic leaves of algebraic foliations over number fields. *Publ. Math. Inst. Hautes Études Sci.*, (93):161–221, 2001.

[8] M. Brunella. Feuilletages holomorphes sur les surfaces complexes compactes. *Ann. Sci. École Norm. Sup. (4)*, 30(5):569–594, 1997.

[9] M. Brunella. Minimal models of foliated algebraic surfaces. *Bull. Soc. Math. France*, 127(2):289–305, 1999.

[10] M. Brunella. *Birational geometry of foliations*. Monografías de Matemática. [Mathematical Monographs]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2000. Available electronically at http://www.impa.br/Publicacoes/Monografias/Abstracts/brunella.ps.

[11] M. Brunella. Foliations on complex projective surfaces. In *Dynamical systems. Part II*, Pubbl. Cent. Ric. Mat. Ennio Giorgi, pages 49–77. Scuola Norm. Sup., Pisa, 2003.

[12] M. Brunella. Subharmonic variation of the leafwise Poincaré metric. *Invent. Math.*, 152(1):119–148, 2003.

[13] S. Druel C. Araujo and S. Kovács. Cohomological characterizations of projective spaces and hyperquadrics. *Invent. Math.*, 174(2):233–253, 2008.

[14] D. Cerveau and A. Lins Neto. Irreducible components of the space of holomorphic foliations of degree two in $\mathbf{C}\mathrm{P}(n)$, $n \geq 3$. *Ann. of Math. (2)*, 143(3):577–612, 1996.

[15] G. Darboux. Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré. *Darboux Bull. (2)*, 2:60–96, 1878.

[16] A. S. de Medeiros. Singular foliations and differential p-forms. *Ann. Fac. Sci. Toulouse Math. (6)*, 9(3):451–466, 2000.

[17] J. Déserti and D. Cerveau. Feuilletages et actions de groupes sur les espaces projectifs. *Mém. Soc. Math. Fr. (N.S.)*, (103):vi+124 pp. (2006), 2005.

[18] J. V. Pereira F. Loray and F. Touzet. Foliations with trivial canonical bundle on Fano 3-folds. *Math. Nachr.*, 286(8-9):921–940, 2013.

[19] J.-P. Jouanolou. *Équations de Pfaff algébriques*, volume 708 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[20] D. Lieberman. Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds. In *Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977)*, volume 670 of *Lecture Notes in Math.*, pages 140–186. Springer, Berlin, 1978.

[21] F. Loray, J.V. Pereira, and F. Touzet. Singular foliations with trivial canonical class, 2011.

[22] M. McQuillan. Canonical models of foliations. *Pure Appl. Math. Q.*, 4(3, part 2):877–1012, 2008.

[23] L. G. Mendes. Kodaira dimension of holomorphic singular foliations. *Bol. Soc. Brasil. Mat. (N.S.)*, 31(2):127–143, 2000.

[24] L.G. Mendes. Bimeromorphic invariants of foliations. *IMPA Phd Thesis*, 1997.

[25] Y. Miyaoka. Deformations of a morphism along a foliation and applications. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 245–268. Amer. Math. Soc., Providence, RI, 1987.

[26] J. V. Pereira and F. Touzet. Foliations with vanishing Chern classes. *Bull. Braz. Math. Soc. (N.S.)*, 44(4):731–754, 2013.

[27] L. Solá Conde S. Kebekus and M. Toma. Rationally connected foliations after Bogomolov and McQuillan. *J. Algebraic Geom.*, 16(1):65–81, 2007.

[28] N.I. Shepherd-Barron. Miyaoka's theorems on the generic seminegativity of T_X and on the Kodaira dimension of minimal regular threefolds. In *Flips and abundance for algebraic threefolds. A summer seminar at the University of Utah, Salt Lake City, 1991*, pages 103–114. Paris: Société Mathématique de France, 1992.

[29] F. Touzet. Feuilletages holomorphes de codimension un dont la classe canonique est triviale. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(4):655–668, 2008.

[30] J. Wahl. A cohomological characterization of $\mathbf{P}^n$. *Invent. Math.*, 72(2):315–322, 1983.