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★**Multiparameter processes. (English summary)**
An introduction to random fields.
Springer Monographs in Mathematics.
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This book presents an updated and comprehensible account on the theory of multiparameter stochastic processes. A multiparameter stochastic process (random field) is a family of random variables $X = \{X_t, t \in \mathbf{T}\}$ indexed by an N-dimensional parameter, which can be discrete ($\mathbf{T} = \mathbf{N}^N$) or continuous ($\mathbf{T} = \mathbf{R}_+^N$). Due to the lack of a useful well-ordering on the parameter set when $N \geq 2$, the usual definitions of martingale, Markov property, and stopping times cannot be extended in a unique and proper way. Therefore, multiple definitions of such notions are possible, giving rise to a richer and more complicated theory. As an illustration of these problems, see the basic reference on stochastic integrals in the plane by R. Cairoli and J. B. Walsh [Acta Math. **134** (1975), 111–183; MR0420845 (54 #8857)].

In the book, one-parameter notions are first studied in detail and the corresponding multiparameter extensions are later introduced in a natural form. This book is in two parts. The first six chapters deal with discrete-time processes, whereas the remaining six chapters treat the continuous-time case. Let us describe the contents of the book.

In Chapter 1, after some basic facts on one-parameter discrete-time martingales, the author studies multiparameter martingales. Following the works by Cairoli and Walsh, two different notions of martingales (and also of submartingales and supermartingales) are analyzed. The first one is related to the behavior of the process on each coordinate (orthomartingales) and the second one is defined in terms of the partial ordering of the parameter set (martingales). The main results are Cairoli's strong and weak inequalities and Cairoli's first and second convergence theorems for orthomartingales. Under the commutation hypothesis it is proved that the two types of martingales coincide. The utility of martingale theory is illustrated in two probabilistic proofs of fundamental analytic theorems given in Chapter 2. The first theorem states that Haar functions form an orthonormal basis for $L^p([0,1])$ for any $p > 1$ and the second is the celebrated Lebesgue differentiation theorem. Some multidimensional extensions of these results are presented using multiparameter martingale methods.

Chapter 3 studies recurrence and transience properties for random walks in the d-dimensional lattice $\mathbf{Z}^d$. An interesting discussion about the intersection of two or many random walks is included. In Chapter 4 an N-parameter random walk on $\mathbf{R}^d$ is defined as $S_t = \sum_{s \leq t} X_s$, where the $\{X_s, s \in \mathbf{N}_+^N\}$ is a family of independent, identically distributed $\mathbf{R}^d$-valued random variables and $s \leq t$ denotes the coordinate-wise ordering. These random walks are analyzed using the tools of multiparameter martingale theory. Several asymptotic results are established. In particular, sufficient conditions are given for the law of large numbers and the law of the iterated logarithm to hold for the N-parameter random walk $\{S_t, t \in \mathbf{N}_+^N\}$, in terms of the law of the increments X_s .

Chapter 5 contains some fundamental results on stochastic processes and Gaussian random variables, including the proofs of Doob's separability theorem and a sharp version of Kolmogorov's continuity theorem. Chapter 6 deals with weak convergence of probability measures on metric spaces with the application of this theory to Donsker's invariance principle.

Chapter 7 deals with continuous parameter martingales. Some basic facts of martingale theory, including the existence of right-continuous modifications for one-parameter martingales, are established. The theory of stochastic integrals of adapted and continuous processes with respect to a continuous square integrable martingale is developed. Stochastic integrals of continuous and adapted processes with respect to the Brownian sheet are also constructed, proving the existence and uniqueness of solutions for nonlinear hyperbolic stochastic partial differential equations.

An introduction to Markov processes is presented in Chapters 8 and 9, including discrete Markov chains, Feller processes, Feynman-Kac's semigroup and the connection of Brownian motion with harmonic functions and the Laplace equation.

The rest of the book is focused on the so-called probabilistic potential theory. The main question addressed by this theory is the following: Given a d-dimensional N-parameter stochastic process $X = \{X_t, t \in \mathbf{R}_+^N\}$, when does the random function X ever enter a given nonrandom set $E \subset \mathbf{R}^d$? In Chapter 10 this problem is studied for one-parameter processes. The starting point is the analysis of transience and recurrence for Lévy processes, based on methods developed by the author. In the case of strongly symmetric Feller processes, some upper and lower bounds are obtained for the Laplace transform of the entrance time on a compact set E, in terms of certain capacities of this set. Explicit computations for the Brownian motion are given.

A class of multiparameter Markov processes with values in a separable locally compact space is introduced and studied in Chapter 11. Examples of these processes are product Feller processes, additive Lévy processes and multiparameter product processes. A potential theory for this class of processes is developed, one of the main results being the characterization of positivity of the intersection (or hitting) probabilities in terms of capacities. Interesting applications to the analysis of the fractal structure of classes of multiparameter processes are provided.

Finally, Chapter 12 is devoted to the probabilistic potential theory for the Brownian sheet, which is not a Markov process in the sense of Chapter 11. This chapter includes recent results by the author on lower and upper bounds for the probability that a d-dimensional N-parameter Brownian sheet intersects a compact set E in terms of the Bessel-Riesz capacity $\mathrm{Cap}_{d-2N}(E)$. An evaluation of the Hausdorff dimension of the zero set of the Brownian sheet and a brief introduction to the theory of local times are also included.

This book is certainly a basic reference for subjects like multiparameter martingales and potential theory for the Brownian sheet and several Markov processes. It can be useful for researchers that would like to learn the basis and recent developments of these subjects. The book is self-contained, except for some marginal results and complements that are left to the reader as exercises. In spite of the technical character of the subject, reading this book is a very pleasant and enriching experience.

Reviewed by *David Nualart*