Math 6070-1, Spring 2006, University of Utah
Solutions to Project #3 [Theoretical Portion]

2. *Suppose $\{X_t\}_{t=1}^{\infty}$ is a stationary time series with mean $\mu := 0$ and autocovariance function γ. Choose and fix two integer times $S < T$. Suppose we wish to estimate X_T based on X_S alone. We wish to use a linear estimator. That is, one of the type $\alpha X_S + \beta$. Find estimates for α and β that are optimal in the sense that they minimize the mean-squared error.*

Solution: Hereforth, let $h := T - S$ denote the lag. The MSE is

$$\begin{aligned} M(\alpha, \beta) &:= E\left[(X_T - \alpha X_S - \beta)^2\right] \\ &= E\left[(X_h - \alpha X_0 - \beta)^2\right], \end{aligned}$$

by stationarity. Because $E(X_t) = 0$, and $\mathrm{Cov}(X_s, X_t) = \gamma(t - s)$ for all s, t, we can expand the previous display to find that

$$\begin{aligned} M(\alpha, \beta) &= E(X_h^2) + \left\{E\left[(\alpha X_0 + \beta)^2\right]\right\} - 2E\left[X_h(\alpha X_0 + \beta)\right] \\ &= \gamma(0) + \alpha^2\gamma(0) + \beta^2 - 2\alpha\gamma(h). \end{aligned}$$

Differentiate to find that

$$\frac{\partial M(\alpha, \beta)}{\partial \alpha} = 2\alpha\gamma(0) - 2\gamma(h) \qquad \text{and} \qquad \frac{\partial M(\alpha, \beta)}{\partial \beta} = 2\beta.$$

Set both to zero to find the "normal equations" for $\hat{\alpha}$ and $\hat{\beta}$. They are:

$$\hat{\alpha} = \frac{\gamma(h)}{\gamma(0)} \quad \text{and} \quad \hat{\beta} = 0.$$

So the best linear estimator of X_T based solely on X_S is

$$\widehat{X_T} := \frac{\gamma(T - S)}{\gamma(0)} X_S.$$

3. *Let $\{W_t\}_{t=-\infty}^{\infty}$ be a white noise process with variance σ^2. Suppose $|\phi| < 1$, and define $X_1 := W_1$, $X_2 := \phi X_1 + W_2$, $X_3 := \phi X_2 + W_3$, $\ldots$, $X_n := \phi X_{n-1} + W_n$ $\ldots$.*

(a) *Prove that $\{X_t\}_{t=1}^{\infty}$ is not stationary. [Hint: It is not even weakly stationary.]*

Solution: **(a)** Recall first that $E(W_t) = 0$, $\text{Cov}(W_s\,, W_t) = 0$ if $s \neq t$, and $\text{Var}(W_t) = \sigma^2$ for all t. Next, we solve for the X_n's:

$$\begin{aligned} X_n &= \phi X_{n-1} + W_n = \phi\left(\phi X_{n-2} + W_{n-1}\right) + W_n \\ &= \phi^2 X_{n-2} + \phi W_{n-1} + W_n \\ &= \phi^2\left(\phi X_{n-3} + W_{n-2}\right) + \phi W_{n-1} + W_n \\ &= \phi^3 X_{n-3} + \phi^2 W_{n-2} + \phi W_{n-1} + W_n \\ &\vdots \\ &= \phi^k X_{n-k} + \sum_{j=0}^{k-1} \phi^j W_{n-j}. \end{aligned}$$

Choose $k := n$ to find that for all $n \geq 1$,

$$X_n = \frac{1}{1-\phi^n} \sum_{j=0}^{n-1} \phi^j W_{n-j}.$$

Consequently, $E(X_n) = 0$ for all $n \geq 1$. But this is a far cry from weak stationarity. For instance, because the W_j's are uncorrelated,

$$\text{Var}(X_n) = \frac{1}{(1-\phi^n)^2} \sum_{j=0}^{n-1} \phi^{2j}\sigma^2 = \frac{1-\phi^{2n}}{(1-\phi^n)^2\,(1-\phi)}\;\sigma^2.$$

As this depends on n, $\{X_n\}_{n=1}^\infty$ cannot be weakly stationary.

(b) *Prove that nonetheless $\{X_t\}_{t=1}^\infty$ is "asymptotically weakly stationary," in the sense that $\gamma_0(h) := \lim_{t\to\infty} \text{Cov}(X_t\,, X_{t+h})$ exists for all $h \geq 0$. Compute the said limit.*

Solution: Because $E(X_n) = 0$, for all $t, h \geq 1$,

$$\text{Cov}(X_t\,, X_{t+h}) = \frac{1}{(1-\phi^t)\,(1-\phi^{t+h})} E\left[\sum_{i=0}^{t-1} \phi^i W_{t-i} \times \sum_{j=0}^{t+h-1} \phi^j W_{t+h-j}\right].$$

But $[W_{t-i}W_{t+h-j}] = 0$ unless $j = i + h$. Therefore,

$$\begin{aligned} \text{Cov}(X_t\,, X_{t+h}) &= \frac{1}{(1-\phi^t)\,(1-\phi^{t+h})} \sum_{i=0}^{t-1} \phi^i \phi^{i+h}\sigma^2 \\ &= \frac{\sigma^2\phi^h}{(1-\phi^t)\,(1-\phi^{t+h})} \sum_{i=0}^{t-1} \phi^{2i} \\ &\longrightarrow \frac{\sigma^2\phi^h}{1-\phi^2} \quad \text{as } t \to \infty. \end{aligned}$$

Thus, $\gamma_0(h) = \sigma^2 \phi^h / (1 - \phi^2)$.

(c) [Hard] *What happens if $|\phi| \geq 1$?*

Solution: First consider the case $\phi = 1$. In this case,

$$X_t = X_{t-1} + W_t = X_{t-2} + W_{t-1} + W_t = \cdots = \sum_{j=1}^{t} W_j.$$

This is obviously not a stationary process. For instance, although $E(X_t) = 0$ for all $t \geq 1$, we have $\text{Var}(X_t) = t\sigma^2$.

If $\phi = -1$, then

$$X_t = -X_{t-1} + W_t = -X_{t-2} - W_{t-1} + W_t = \cdots = \sum_{j=1}^{t} (-1)^j W_j.$$

Therefore, $\text{Var}(X_t) = t\sigma^2$, whence follows the non-stationarity of $\{X_t\}_{t=1}^{\infty}$.

If $|\phi| > 1$, then one can still solve for X_t explicitly. But we can simply note that

$$X_t^2 = \phi^2 X_{t-1}^2 + W_t^2 + 2\phi X_{t-1} W_t,$$

and so (why?),

$$\begin{aligned}\text{Var}(X_t) = E(X_t^2) &= \phi^2 E(X_{t-1}^2) + \sigma^2 \geq \phi^2 E(X_{t-1}^2) \\ &\geq \phi^4 E(X_{t-2}^2) \geq \cdots \phi^{2k} E(X_{t-k}^2) \geq \cdots \\ &\geq \phi^{2(t-1)} \sigma^2.\end{aligned}$$

Because $|\phi| > 1$, this proves that $\lim_{t \to \infty} \text{Var}(X_t) = \infty$. In particular, $\{X_t\}_{t=1}^{\infty}$ cannot be weakly stationary.