Homework #3
Math 6070-1, Spring 2006

1. Consider a probability density kernel K of the form

$$K(x) = \frac{1}{2\tau} e^{-|x|/\tau}, \qquad -\infty < x < \infty.$$

Here, $\tau > 0$ is fixed. Assuming that f is sufficiently smooth, then derive the form of the asymptotically optimal bandwidth h_n in the same manner as we did in the lectures for the case $\tau = 1$. The extra parameter τ is often used to refine kernel-density estimates that are based on the double-exponential family.

2. Construct continuous probability densities $f_1, f_2, \ldots$ and f such that:

   (a) $\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx = 0$; and

   (b) there exist infinitely-many values of x such that $f_n(x) \not\to f(x)$ as $n \to \infty$.

   Thus, convergence in L^1 is not the same as ordinary (pointwise) convergence.

3. Prove that if f and g are probability densities, then $\mathscr{F}(f*g)(t) = (\mathscr{F}f)(t) \times (\mathscr{F}g)(t)$ for all t. Use this to prove that if f is a probability density and ϕ_ϵ is the $N(0, \epsilon^2)$ density, then $f * \phi_\epsilon$ has an integrable Fourier transform.

4. Let $X_1, X_2, \ldots$ be an i.i.d. sample from a density function f. We assume that f is differentiable in an open neighborhood V of a fixed point x, and $B := \max_{z\in V} |f'(z)| < \infty$.

   (a) Prove that for all $\lambda > 0$, $m \geq 1$, and all $x \in \mathbf{R}$,

$$\mathrm{P}\left\{ \min_{1\leq j\leq m} |X_j - x| \geq \lambda \right\} = \left[1 - \int_{x-\lambda}^{x+\lambda} f(z)\, dz \right]^m.$$

   (b) Prove that for all $\epsilon > 0$ small enough,

$$\max_{z\in[x-\epsilon\, , x+\epsilon]} \left| f(x) - f(z) \right| \leq 2B\epsilon.$$

   Use this to estimate $| \int_{x-\epsilon}^{x+\epsilon} f(z)\, dz - 2\epsilon f(x)|$.

   (c) Suppose that as $m \to \infty$, $\lambda_m \to \infty$ and $\lambda_m^2/m \to 0$. Then, prove that

$$\lim_{m\to\infty} \frac{-1}{2\lambda_m} \ln \mathrm{P}\left\{ \min_{1\leq j\leq m} |X_j - x| \geq \frac{\lambda_m}{m} \right\} = f(x).$$

   (d) Devise an estimator of $f(x)$ based on the previous steps.