Homework #2
Math 6070-1, Spring 2006

Let $X_1, X_2, \ldots, X_n$ be random variables, all i.i.d., and with the same distribution function F that has density $f := F'$. Define for all $p \geq 1$,

$$Q_n^{(p)}(F) := \left\{ \int_{-\infty}^{\infty} \left| \hat{F}_n(x) - F(x) \right|^p f(x)\, dx \right\}^{1/p}.$$

This is a kind of "distance" between $\hat{F}_n$ and F, although it is different from $D_n(F)$.

1. Prove that $Q_n^{(p)}(F) \leq 2$, so $Q_n^{(p)}(F)$ is always finite.

2. Compute $Q_n^{(p)}(F_u)$ where $F_u(x) = x$ for $0 \leq x \leq 1$, $F_u(x) = 0$ if $x < 0$, and $F_u(x) = 1$ if $x \geq 1$.

3. Prove that $Q_n^{(p)}(F)$ is distribution-free. That is, if F and G are strictly increasing distribution functions such that F' and G' both exist, then the distribution of $Q_n^{(p)}(F)$ is the same as that of $Q_n^{(p)}(G)$.

4. Prove that under F, as $n \to \infty$, $Q_n^{(p)}(F) \xrightarrow{\text{P}} 0$, provided that F is strictly increasing and has a density. Do this by first proving that

$$\mathrm{E}_F\left[\left| Q_n^{(2)}(F) \right|^2 \right] = \frac{1}{6n}.$$

5. Suppose F is strictly increasing has a density. Then provide a heuristic justification of the fact that, under F, as $n \to \infty$,

$$\sqrt{n}\, Q_n^{(p)}(F) \xrightarrow{d} \left\{ \int_0^1 |B^\circ(x)|^p\, dx \right\}^{1/p},$$

   where B° denotes the Brownian bridge on $[0\,,1]$. Later on in a Project we will see how to simulate the distribution of the latter limiting object.

6. Use 4 to test $H_0:\ F = F_0$ versus $H_1:\ F \neq F_0$ for a known distribution function F_0 that is strictly increasing and has a density.

7. Suppose F has a density f which satisfies $f(x) > 0$ for all x. Then prove that F is strictly increasing.