Homework #1
Math 6070-1, Spring 2006

1. Compute, carefully, the moment generating function of a Gamma$(\alpha\,,\beta)$. Use it to compute the moments of a Gamma-distributed random variable.

2. Let $X_1, X_2, \ldots, X_n$ be an independent sample (i.e., they are i.i.d.) with finite mean $\mu = \mathrm{E}X_1$ and variance $\sigma^2 = \mathrm{Var}X_1$. Define

$$\hat{\sigma}_n^2 := \frac{1}{n}\sum_{j=1}^{n}\left(X_j - \bar{X}_n\right)^2, \tag{1}$$

where $\bar{X}_n := (X_1 + \cdots + X_n)/n$ denotes the sample average. First, compute $\mathrm{E}\hat{\sigma}_n^2$. Then prove, carefully, that $\hat{\sigma}_n^2$ converges in probability to σ^2.

3. Let U have the Uniform $(0\,,1)$ distribution.

   (a) Prove that if F is a distribution function and F^{-1}—its inverse function—exists, then the distribution function of $X := F^{-1}(U)$ is F.

   (b) Use the preceding to prove that $X := \tan U$ has the Cauchy distribution. That is, the density function of Y is

$$f_X(a) := \frac{1}{\pi(1+a^2)}, \qquad -\infty < a < \infty. \tag{2}$$

   (c) Use the preceding to find a function h such that $Y := h(U)$ has the Expontential (λ) distribution.

4. A random variable X has the *logistic* distribution if its density function is

$$f(x) = \frac{e^{-x}}{\left(1+e^{-x}\right)^2}, \qquad -\infty < x < \infty. \tag{3}$$

   (a) Compute the distribution function of X.

   (b) Compute the moment generating function of X. (Hint: The answer is in terms of gamma functions.)

   (c) Prove that $\mathrm{E}\{|X|^r\} < \infty$ for all $r > 0$.