Solutions to Homework #3
Math 6070-1, Spring 2006

1. *Consider a probability density kernel K of the form*

$$K(x) = \frac{1}{2\tau} e^{-|x|/\tau}, \qquad -\infty < x < \infty.$$

Here, $\tau > 0$ is fixed. Assuming that f is sufficiently smooth, then derive the form of the asymptotically optimal bandwidth h_n in the same manner as we did in the lectures for the case $\tau = 1$. The extra parameter τ is often used to refine kernel-density estimates that are based on the double-exponential family.

Solution: We compute β_K, σ_K^2, and $\|K\|_2^2$ and plug.

Note that

$$\|K\|_2^2 = \frac{1}{4\tau^2} \int_{-\infty}^{\infty} e^{-2|x|/\tau}\, dx = \frac{1}{4\tau}.$$

Next, we have

$$\sigma_K^2 = \frac{1}{2\tau} \int_{-\infty}^{\infty} x^2 e^{-|x|/\tau}\, dx = \tau^2 \int_0^{\infty} x^2 e^{-x}\, dx = \Gamma(3)\tau^2 = 2\tau^2.$$

Finally,

$$\beta_K = \frac{\|K\|_2^{2/5}}{\sigma_K^{4/5}} = \frac{(1/(4\tau))^{1/5}}{(2\tau^2)^{2/5}} = \frac{1}{16^{1/5}\tau}.$$

2. *Construct continuous probability densities $f_1, f_2, \ldots$ and f such that:*

   (a) $\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx = 0$; *and*

   (b) *there exist infinitely-many values of x such that $f_n(x) \not\to f(x)$ as $n \to \infty$.*

Solution: I will do a little more and produce a "fancy" counter-example. Namely, I will construct examples of densities $f, f_1, f_2, \ldots$ such that: (i) $\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx = 0$; and yet (ii) for all rational numbers r, $\lim_{n\to\infty} |f_n(r) - f(r)| = \infty$.

Because rationals are countable we can label them. So let $\mu_1, \mu_2, \ldots$ be an enumeration of all rational numbers. Also, let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that: (i) $1/\epsilon_n^{1/4}$ is a positive integer; and (ii) $\lim_{n\to\infty} \epsilon_n = 0$. For example, you can set $\epsilon_n := n^{-4}$ if you would like. Define for all $i, n \geq 1$,

$$\phi_{i,n}(x) := \frac{1}{\epsilon_n \sqrt{2\pi}} \exp\left(-\frac{(x-\mu_i)^2}{2\epsilon_n^2} \right).$$

Define f to be an arbitrary density such that $0 \leq f(x) < \infty$ for all $x \in \mathbf{R}$. And for all $n \geq 1$ define

$$f_n(x) = \left(1 - \epsilon_n^{1/4}\right) f(x) + \epsilon_n^{1/2} \sum_{i=1}^{1/\epsilon_n^{1/4}} \phi_{i,n}(x).$$

Check that $f_n(x) \geq 0$ and $\int_{-\infty}^{\infty} f_n(x)\, dx = 1$. So each f_n is a density function. Moreover, the triangle inequality implies that

$$|f_n(x) - f(x)| \leq \epsilon_n^{1/4} f(x) + \epsilon_n^{1/2} \sum_{i=1}^{1/\epsilon_n^{1/4}} \phi_{i,n}(x).$$

Therefore,

$$\int_{-\infty}^{\infty} |f_n(x) - f(x)|\, dx \leq 2\epsilon_n^{1/4} \to 0 \quad \text{as } n \to \infty.$$

But we can note that for all i fixed, if $n \geq i$ then

$$f_n(\mu_i) \geq \epsilon_n^{1/2} \phi_{i,n}(\mu_i) = \frac{1}{\sqrt{2\pi\epsilon_n}} \to \infty, \quad \text{as } n \to \infty.$$

3. *Prove that if f and g are probability densities, then $\mathscr{F}(f*g)(t) = (\mathscr{F}f)(t) \times (\mathscr{F}g)(t)$ for all t. Use this to prove that if f is a probability density and ϕ_ϵ is the $N(0\,, \epsilon^2)$ density, then $f * \phi_\epsilon$ has an integrable Fourier transform.*

   Solution: Compute directly to find that

$$\begin{aligned}
\mathscr{F}(f*g)(t) &= \int_{-\infty}^{\infty} e^{itx}(f*g)(x)\, dx \\
&= \int_{-\infty}^{\infty} e^{itx} \left(\int_{-\infty}^{\infty} f(y) g(x-y)\, dy \right) dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{it(x-y)} g(x-y)\, dx \right) e^{ity} f(y)\, dy \\
&= (\mathscr{F}f)(t) \times (\mathscr{F}g)(t).
\end{aligned}$$

   Recall that $(\mathscr{F}\phi_\epsilon)(t) = e^{-t^2\epsilon^2/2}$. Because $|(\mathscr{F}f)(t)| \leq 1$, it follows that $|\mathscr{F}(f*\phi_\epsilon)(t)| \leq e^{-t^2\epsilon^2/2}$, which is integrable. In fact,

$$\int_{-\infty}^{\infty} |\mathscr{F}(f*\phi_\epsilon)(t)|\, dt \leq \int_{-\infty}^{\infty} e^{-t^2\epsilon^2/2}\, dt = \frac{\sqrt{2\pi}}{\epsilon} < \infty.$$

4. *Let $X_1, X_2, \ldots$ be an i.i.d. sample from a density function f. We assume that f is differentiable in an open neighborhood V of a fixed point x, and $B := \max_{z \in V} |f'(z)| < \infty$.*

(a) *Prove that for all* $\lambda > 0$, $m \geq 1$, *and all* $x \in \mathbf{R}$,

$$\mathrm{P}\left\{\min_{1\leq j\leq m} |X_j - x| \geq \lambda\right\} = \left[1 - \int_{x-\lambda}^{x+\lambda} f(z)\, dz\right]^m.$$

Solution: By independence,

$$\mathrm{P}\left\{\min_{1\leq j\leq m} |X_j - x| \geq \lambda\right\} = \left[\mathrm{P}\left\{|X_1 - x| \geq \lambda\right\}\right]^m.$$

Compute the latter probability to finish.

(b) *Prove that for all* $\epsilon > 0$ *small enough,*

$$\max_{z\in[x-\epsilon,x+\epsilon]} \left|f(x) - f(z)\right| \leq 2B\epsilon.$$

Use this to estimate $|\int_{x-\epsilon}^{x+\epsilon} f(z)\, dz - 2\epsilon f(x)|$.

Solution: Apply Taylor's expansion with remainder to find that $|f(x) - f(z)| \leq B|x - z|$ whenever $z \in V$. This implies the first claim. For the second claim we note that by the triangle inequality,

$$\left|\int_{x-\epsilon}^{x+\epsilon} f(z)\, dz - 2\epsilon f(x)\right| \leq \int_{x-\epsilon}^{x+\epsilon} |f(z) - f(x)|\, dx.$$

Apply the first claim to deduce that this is at most $4B\epsilon^2$.

(c) *Suppose that as* $m \to \infty$, $\lambda_m \to \infty$ *and* $\lambda_m^2/m \to 0$. *Then, prove that*

$$\lim_{m\to\infty} \frac{-1}{2\lambda_m} \ln \mathrm{P}\left\{\min_{1\leq j\leq m} |X_j - x| \geq \frac{\lambda_m}{m}\right\} = f(x).$$

Solution: According to part (a) we can write

$$\mathrm{P}\left\{\min_{1\leq j\leq m} |X_j - x| \geq \frac{\lambda_m}{m}\right\} = \left[1 - \int_{x-(\lambda_m/m)}^{x+(\lambda_m/m)} f(z)\, dz\right]^m.$$

Apply (b) to find that

$$\mathrm{P}\left\{\min_{1\leq j\leq m} |X_j - x| \geq \frac{\lambda_m}{m}\right\} = \left[1 - \frac{2\lambda_m}{m} f(x) + \delta_m\right]^m,$$

where $|\delta_m| \leq 4B(\lambda_m/m)^2$. Now recall that for $\eta > 0$ small,

$$\ln(1 - \eta) = -\eta + \frac{1}{2}\eta^2 - \frac{1}{3}\eta^3 \pm \cdots.$$

Thus,

$$\left[1 - \frac{2\lambda_m}{m} f(x) + \delta_m\right]^m = e^{-2\lambda_m f(x) + m\delta_m \pm \cdots}.$$

Because $\lambda_m^2/m \to 0$, it follows that $(m\delta_m + \cdots) \to 0$ as $m \to \infty$. The assertion follows.

(d) *Devise an estimator of $f(x)$ based on the previous steps.*

Solution: Choose and fix an enormous integer $m \geq 1$. Our goal is to estimate

$$\theta_m := \frac{-1}{2\lambda_m} \ln \mathrm{P}\left\{ \min_{1\leq j\leq m} |X_j - x| \geq \frac{\lambda_m}{m} \right\}.$$

Choose and fix a massively large integer $n \geq 1$. Simulate n copies of $(X_1, \ldots, X_m)$. Say $(X_1^1, \ldots, X_m^1)$ through $(X_1^n, \ldots, X_m^n)$. Then by the law of large numbers, as $n \to \infty$,

$$\hat{\theta}_{m,n} := \frac{-1}{2\lambda_m} \ln\left(\frac{1}{n} \sum_{\ell=1}^{n} \mathrm{I}\left\{ \min_{1\leq j\leq m} \left|X_j^\ell - x\right| \geq \frac{\lambda_m}{m} \right\} \right) \xrightarrow{\mathrm{P}} \theta_m.$$