Solutions to Homework #1
Math 6070-1, Spring 2006

1. *Compute, carefully, the moment generating function of a Gamma$(\alpha\,,\beta)$. Use it to compute the moments of a Gamma-distributed random variable.*

   Solution: Recall that

   $$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \qquad 0 < x < \infty.$$

   Therefore,

   $$M(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x}\, dx.$$

   If $t \geq \beta$, then $M(t) = \infty$. But if $t < \beta$ then $M(t) = \beta^\alpha/(\beta-t)^\alpha$. Its derivatives are $M'(t) = \beta^\alpha \alpha(\beta-t)^{-\alpha-1}$, $M''(t) = \beta^\alpha\alpha(\alpha-1)(\beta-t)^{-\alpha-2}$, and so on. The general term is $M^{(k)}(t) = \beta^\alpha(\beta-t)^{-\alpha-k}\prod_{\ell=0}^{k-1}(\alpha-\ell)$. Thus, $\mathrm{E}X = M'(0) = \alpha/\beta$, $\mathrm{E}X^2 = M''(0) = \alpha(\alpha-1)/\beta^2$, and so. In general, we have $\mathrm{E}X^k = \beta^{-k}\prod_{j=0}^{k-1}(\alpha-j)$.

2. *Let $X_1, X_2, \ldots, X_n$ be an independent sample (i.e., they are i.i.d.) with finite mean $\mu = \mathrm{E}X_1$ and variance $\sigma^2 = \mathrm{Var}X_1$. Define*

   $$\hat\sigma_n^2 := \frac{1}{n}\sum_{j=1}^n \left(X_j - \bar X_n\right)^2, \tag{1}$$

   where $\bar X_n := (X_1+\cdots+X_n)/n$ denotes the sample average. First, compute $\mathrm{E}\hat\sigma_n^2$. Then prove, carefully, that $\hat\sigma_n^2$ converges in probability to σ^2.

   Solution: $\hat\sigma_n^2 = n^{-1}\sum_{j=1}^n X_j^2 - (\bar X_n)^2$. Therefore, $\mathrm{E}\hat\sigma_n^2 = \mathrm{E}[X_1^2] - \mathrm{E}[(\bar X_n)^2]$. But $\mathrm{E}X_1^2 = \mathrm{Var}(X_1) + \mu^2 = \sigma^2 + \mu^2$. Similarly, $\mathrm{E}[(\bar X_n)^2] = \mathrm{Var}(\bar X_n) + (\mathrm{E}\bar X_n)^2 = (\sigma^2/n) + \mu^2$. Therefore, $\mathrm{E}\hat\sigma_n^2 = \sigma^2(n-1)/n$. As regards the large-sample theory ... we apply the law of large numbers twice: $\bar X_n \xrightarrow{\mathrm{P}} \mu$; and $n^{-1}\sum_{=1}^n X_i^2 \xrightarrow{\mathrm{P}} \sigma^2+\mu^2$. This proves that $\hat\sigma_n^2$ is a consistent estimator of σ^2.

3. *Let U have the Uniform-$(0\,,1)$ distribution.*

   (a) *Prove that if F is a distribution function and F^{-1}—its inverse function—exists, then the distribution function of $X := F^{-1}(U)$ is F.*

   Solution: $\mathrm{P}\{X \leq x\} = \mathrm{P}\{U \leq F(x)\} = F(x)$, whence we have $F_X = F$, as desired.

   (b) *Use the preceding to prove that $X := \tan U$ has the Cauchy distribution. That is, the density function of Y is*

   $$f_X(a) := \frac{1}{\pi(1+a^2)}, \qquad -\infty < a < \infty. \tag{2}$$

Solution and Correction: Let C have the Cauchy distribution. Then,

$$\begin{aligned} F_C(a) = \mathrm{P}\{C \le a\} &= \int_{-\infty}^{a} \frac{du}{\pi(1+u^2)} \\ &= \frac{1}{\pi}\arctan a - \frac{1}{\pi}\arctan(-\infty) = \frac{1}{\pi}\arctan(a) + \frac{1}{2}. \end{aligned}$$

Therefore, $F_C^{-1}(a) = \arctan(\pi x - \frac{\pi}{2})$, and $F_C^{-1}(U) \sim$ Cauchy. Note that $V := \pi U - (\pi/2) \sim$ Uniform-$(-\pi/2\,,\pi/2)$. So $\arctan(V)$ is indeed Cauchy, where $V \sim$ Uniform-$(-\pi/2\,,\pi/2)$.

(c) *Use the preceding to find a function h such that $Y := h(U)$ has the Exponential (λ) distribution.*

Solution: If $X \sim \mathrm{Exp}(\lambda)$ then $F(x) = \int_{-\infty}^{x} \lambda e^{-\lambda z}\,dz = 1 - e^{-\lambda x}$. Thus, $h(x) := F^{-1}(x) = -\lambda^{-1}\ln(1-x)$, and $F^{-1}(U) \sim \mathrm{Exp}(\lambda)$. Note that $1-U$ has the same distribution as U. So $-\lambda^{-1}\ln(U) \sim$ Exponential(λ).

4. *A random variable X has the* logistic *distribution if its density function is*

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \qquad -\infty < x < \infty. \tag{3}$$

(a) *Compute the distribution function of X.*

Solution: Set $y := 1 + e^{-x}$ to find that

$$\begin{aligned} F(a) &= \int_{-\infty}^{a} \frac{e^{-x}}{(1+e^{-x})^2}\,dx = \int_{1+\exp(-a)}^{\infty} \frac{dy}{y^2} \\ &= \frac{1}{1+e^{-a}}. \end{aligned}$$

(b) *Compute the moment generating function of X.*

Solution: Again set $y := e^{-x}$ to find that

$$M(t) = \int_{-\infty}^{\infty} \frac{e^{(t-1)x}}{(1+e^{-x})^2}\,dx = \int_0^{\infty} \frac{y^{-t}}{(1+y)^2}\,dy.$$

If $|t| \ge 1$ then this is ∞ [consider the integral for small y when $t \ge 1$, or large y when $t \le -1$]. On the other hand, if $|t| < 1$ then this is finite. Remarkably enough, $M(t)$ can be computed explicitly when $|t| < 1$. Note that

$$\frac{1}{(1+y)^2} = \int_0^{\infty} z e^{-z(1+y)}\,dz.$$

Therefore,

$$M(t) = \int_0^\infty \left(\int_0^\infty z e^{-z(1+y)}\, dz \right) y^{-t}\, dy$$
$$= \int_0^\infty \left(\int_0^\infty y^{-t} e^{-zy}\, dy \right) z e^{-z}\, dz,$$

because the order of integration can be reversed. A change of variable $[w := yz]$ shows that the inner integral is

$$\int_0^\infty y^{-t} e^{-zy}\, dy = z^{t-1} \int_0^\infty w^{-t} e^{-w}\, dw$$
$$= z^{t-1}\Gamma(1-t).$$

Therefore, whenever $|t| < 1$,

$$M(t) = \Gamma(1-t) \int_0^\infty z^t e^{-z}\, dz$$
$$= \Gamma(1-t)\Gamma(1+t).$$

There are other ways of deriving this identity as well. For instance, we can write $w := (1+y)^{-1}$, so that $dw = -(1+y)^{-2}\, dy$ and $y = (1/w) - 1$. Thus,

$$M(t) = \int_0^1 \left(\frac{1}{w} - 1 \right)^{-t} dw$$
$$= \int_0^1 w^t (1-w)^{-t}\, dw = \mathrm{B}(1+t\,, 1-t),$$

where $\mathrm{B}(a\,, b) := \int_0^1 w^{a-1}(1-w)^{b-1}\, dw$ denotes the "beta function." From tables (for instance), we know that $\mathrm{B}(a\,, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Therefore, $M(t) = \Gamma(1+t)\Gamma(1-t)/\Gamma(2) = \Gamma(1+t)\Gamma(1-t)$, as desired.

(c) *Prove that* $\mathrm{E}\{|X|^r\} < \infty$ *for all* $r > 0$.

Solution: We will prove a more general result: If $M(t) < \infty$ for all $t \in (-t_0\,, t_0)$ where $t_0 > 0$, then $\mathrm{E}(|X|^k) < \infty$ for all $k \geq 1$. Recall that $\mathrm{E}Z = \int_0^\infty \mathrm{P}\{Z > t\}\, dt$ whenever $Z \geq 0$. Apply this with $Z := |X|^k$ to find that

$$\mathrm{E}(|X|^k) = \int_0^\infty \mathrm{P}\left\{ |X|^k > r \right\} dr$$
$$= \int_0^\infty \mathrm{P}\left\{ |X| > r^{1/k} \right\} dr$$
$$= k \int_0^\infty \mathrm{P}\{|X| > s\} s^{k-1}\, ds.$$

[Set $s := r^{1/k}$.] Fix $t \in (-t_0\,, t_0)$. Then, by Chebyshev's inequality,

$$\mathrm{P}\{|X| > s\} = \mathrm{P}\left\{e^{t|X|} \geq e^{ts}\right\} \leq \frac{\mathrm{E}[e^{t|X|}]}{e^{ts}}.$$

But $e^{t|x|} \leq e^{tx} + e^{-tx}$ for all $x \in \mathbf{R}$, with room to spare. Therefore, $\mathrm{E}[r^{t|X|}] \leq M(t) + M(-t)$, whence it follows that for $t \in (-t_0\,, t_0)$ fixed and $A := M(t) + M(-t)$, $\mathrm{P}\{|X| > s\} \leq Ae^{-st}$. Thus,

$$\mathrm{E}(|X|^k) \leq Ak \int_0^\infty s^{k-1} e^{-st}\, ds = Akt^k \Gamma(k) = Ak!t^k < \infty.$$