9.1. If X and Y are independent, then for all bounded, continuous $f, g : \mathbf{R}^d \to \mathbf{R}$,

$$\mathrm{E}[f(X)g(Y)] = \mathrm{E}[f(X)] \cdot \mathrm{E}[g(Y)].$$

Apply this with $f(x) = \exp(iu \cdot x)$ and $g(y) = \exp(iv \cdot y)$ to obtain half of the result. Conversely, suppose that the preceding identity holds for the stated f and g, for all such u and v. By the inversion theorem, for all $\psi \in L^1(\mathrm{P})$,

$$\psi(x) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{-iu \cdot x} \widehat{\psi}(u)\, du.$$

Therefore, for all $\psi, \phi \in L^1(\mathrm{P})$, $\mathrm{E}[\psi(X)\phi(Y)] = \mathrm{E}[\psi(X)]\mathrm{E}[\phi(Y)]$, thanks to Fubini–Tonelli.

To derive Corollary 9.7 note that the covariance matrix of $(X_1, \ldots, X_n, Y_1, \cdots, Y_m)$ has the form

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix},$$

where Q_1 is the covariance matrix of $(X_1, \ldots, X_n)$ and Q_2 is that of $(Y_1, \ldots, Y_m)$. In particular, for all $\alpha \in \mathbf{R}^n$ and $\beta \in \mathbf{R}^m$,

$$\begin{aligned} \mathrm{E}\left[e^{i(\alpha,\beta)\cdot(X_1,\ldots,X_n,Y_1,\ldots,Y_m)}\right] &= \exp\left(-\frac{1}{2}(\alpha,\beta) \cdot Q(\alpha,\beta)\right) \\ &= \exp\left(-\frac{1}{2}\alpha \cdot Q_1\alpha - \frac{1}{2}\beta \cdot Q_2\beta\right) \\ &= \mathrm{E}\left[e^{i\alpha\cdot(X_1,\ldots,X_n)}\right] \times \mathrm{E}\left[e^{i\beta\cdot(Y_1,\ldots,Y_m)}\right]. \end{aligned}$$

The asserted independence follows from the first part.