8.8. Define $f_{X|Y}(x|y) = f(x,y)/f_Y(y)$ where $f_Y(y) = \int_{-\infty}^{\infty} f(a,y)\,da$; this shows that our goal is to prove that $\mathrm{E}[h(X)\,|\,Y] = \int_{-\infty}^{\infty} h(x) f_{X|Y}(x|Y)\,dx$ a.s., which is the classical formula on $\{Y = y\}$.

For any positive, bounded function g,

$$\begin{aligned} \mathrm{E}\left[h(X)g(Y)\right] &= \int_{-\infty}^{\infty} g(y) \left(\int_{-\infty}^{\infty} h(x) f(x,y)\,dx \right) dy \\ &= \int_{-\infty}^{\infty} g(y) f_Y(y) \left(\int_{-\infty}^{\infty} h(x) f_{X|Y}(x|y)\,dx \right) dy \\ &= \mathrm{E}\left[\Pi(Y)g(Y)\right], \end{aligned} \tag{8.2}$$

where $\Pi(y) = \int_{-\infty}^{\infty} h(x) f_{X|Y}(x|y)\,dx$. By a monotone class argument, for all bounded $\sigma(Y)$-measurable Z, $\mathrm{E}[h(X)Z] = \mathrm{E}[\Pi(Y)Z]$. Therefore, $\mathrm{E}[h(X)\,|\,Y] = \Pi(Y)$ a.s., which is the desired result.

Apply (8.2) with $h(x) = \mathbf{1}_{(-\infty,a]}(x)$ to find that $\mathrm{P}(X \le a\,|\,Y) = \int_{-\infty}^{a} f_{X|Y}(u|y)\,du$, as desired.