8.2. Let X, Y, and Z be three independent random variables, each taking the values ± 1 with probability $\frac{1}{2}$ each. Define $W = X$, $U = X + Y$, and $V = X + Z$. Then, $\mathrm{E}[U \,|\, W] = X + \mathrm{E}[Y] = X$ a.s. In particular,

$$\begin{aligned} \mathrm{E}\left[\mathrm{E}(U \,|\, W) \,|\, V\right] = \mathrm{E}[X \,|\, V] &= 1 \quad \text{a.s. on } \{V = 2\} \\ &= -1 \quad \text{a.s. on } \{V = -2\}. \end{aligned} \tag{8.1}$$

On the other hand, $\mathrm{E}[X; V = 0] = \mathrm{E}[X; X = 1, Z = -1] + \mathrm{E}[X; X = -1, Z = 1] = 0$. Therefore, to summarize: $\mathrm{E}\{\mathrm{E}(U \,|\, W) \,|\, V\} = V/2$ a.s. A similar analysis shows that $\mathrm{E}[U \,|\, V] = V/2$ a.s also. Thus,

$$\mathrm{E}\left[\mathrm{E}(U \,|\, V) \,|\, W\right] = \frac{1}{2}\mathrm{E}[V \,|\, W] = \frac{1}{2}X \qquad \text{a.s.}$$

Because $\mathrm{P}\{V \neq X\} = 1$, we have produced an example wherein $\mathrm{E}[\mathrm{E}(U \,|\, V) \,|\, W] \neq \mathrm{E}[\mathrm{E}(U \,|\, W) \,|\, V]$ a.s.