**8.11.** We may use the elementary fact that $S$ is a stopping time iff $\{S \le n\} \in \mathscr{F}_n$ for all $n \ge 1$. This is equivalent to: $\{S > n\} \in \mathscr{F}_n$ for all $n \ge 1$.

By induction, we need only consider the case $n = 2$. For all $t \ge 0$, $\{T_1 + T_2 = n\} = \cup_{k=1}^{n}(\{T_1 = k\} \cap \{T_2 = n-k\}) \in \mathscr{F}_n$. Therefore, $T_1 + T_2$ is a stopping time. Also, $\{\min(T_1, T_2) > n\} = \{T_1 > n\} \cap \{T_2 > n\}$. Therefore, $\min(T_1, T_2)$ is a stopping time. Finally, $\{\max(T_1, T_2) \le n\} = \{T_1 \le n\} \cap \{T_2 \le n\}$, and so $\max(T_1, T_2)$ is a stopping time too. This proves Lemma 8.27. Next we prove Lemma 8.28.

If $A \in \mathscr{F}_S$ then for all $n \ge 1$, $A \cap \{T = n\} = \cup_{m=1}^{m} A \cap \{T = n\} \cap \{S = m\}$. By the definition of $\mathscr{F}_S$, $A \cap \{S = m\} \in \mathscr{F}_m \subset \mathscr{F}_n$. Therefore, $A \cap \{S = m\} \cap \{T = m\} \in \mathscr{F}_n$, which means that $A \in \mathscr{F}_T$. Therefore, $\mathscr{F}_S \subseteq \mathscr{F}_T$.

To prove that $\mathscr{F}_T$ is a $\sigma$-algebra, let $A \in \mathscr{F}_T$. We know that $\{T = n\} \in \mathscr{F}_n$ and $A \cap \{T = n\}$ are in $\mathscr{F}_n$ for all $n \ge 1$. Therefore, so is $A^c \cap \{T = n\} = \{T = n\} \cap (A \cap \{T = n\})^c$. This proves that $\mathscr{F}_T$ is closed under complementation. If $A_1, A_2, \ldots \in \mathscr{F}_T$ then $\cup_{i=1}^{\infty} A_i \cap \{T = n\} = \cup_{i=1}^{n}(A_i \cap \{T = n\}) \in \mathscr{F}_n$ for all $n$. Therefore, $\mathscr{F}_T$ is a $\sigma$-algebra.

So far, we needed $S$ and $T$ to be a.s. finite only. For the remaining assertions we assume that $T$ is a.s. bounded—say $T \le k$ a.s.; it suffices to consider a submartingale $X$. Let $d_1 = X_1$ and $d_j = X_j - X_{j-1}$ ($j \ge 2$). For all $A \in \mathscr{F}_S$,

$$\begin{aligned}\mathrm{E}\left[X_T - X_S; A\right] &= \mathrm{E}\left[\sum_{j=S+1}^{T} d_j \,;\, A\right] = \mathrm{E}\left[\sum_{j=1}^{k} \mathbf{1}_{\{S<j\le T\}\cap A} d_j\right] \\ &= \sum_{j=1}^{k} \mathrm{E}\left[\mathbf{1}_{\{S<j\le T\}\cap A} \mathrm{E}(d_j \,|\, \mathscr{F}_{j-1})\right] \ge 0,\end{aligned}$$

because $\{S < j \le T\} \cap A = A \cap \{S \le j-1\}^c \cap \{T \le j-1\}^c \in \mathscr{F}_{j-1}$.