

7.5. [We need to know also that S and D have the same variance.] Let $S = X + Y$ and $D = X - Y$. Note that $X = (S + D)/2$ and $Y = (S - D)/2$. Thus,

$$\mathbf{E} \left[e^{itX + isY} \right] = \mathbf{E} \left[2^{it(S+D)/2 + is(S-D)/2} \right] = \mathbf{E} \left[e^{i(t+s)S/2} \right] \mathbf{E} \left[e^{i(t-s)D/2} \right],$$

by the independent of S and D . Now suppose S is $N(\mu, \sigma^2)$ and D is $N(\nu, \sigma^2)$. Then,

$$\begin{aligned} \mathbf{E} \left[e^{itX + isY} \right] &= e^{i(t+s)\mu/2} e^{i(t-s)\nu/2} e^{-(t+s)^2\sigma^2/8} e^{-(t-s)^2\sigma^2/8} \\ &= e^{\frac{1}{2}it(\mu+\nu) - \frac{1}{8}t^2(\sigma^2+\sigma^2)} e^{\frac{1}{2}is(\mu-\nu) - \frac{1}{8}s^2(\sigma^2+\sigma^2)}, \end{aligned}$$

which is the desired result.