

7.33. $P\{T_n > k\} = P\{X_1 + \cdots + X_k < n\}$. Now,

$$E \left[\exp \left(i \frac{\xi}{n} \sum_{j=1}^k X_j \right) \right] = (E[\exp(i\xi X_1/n)])^k = \left(\frac{1}{n} \sum_{j=1}^n e^{ij\xi/n} \right)^k.$$

Fix $k \geq 1$ and let $n \rightarrow \infty$ to find that

$$E \left[\exp \left(i \frac{\xi}{n} \sum_{j=1}^k X_j \right) \right] \rightarrow \left(\int_0^1 e^{it\xi} dt \right)^k = E \left[\exp \left(i\xi \sum_{j=1}^k U_j \right) \right],$$

where U_1, U_2, \dots, U_k are i.i.d. $\text{Unif}(0, 1)$'s. By the convergence theorem of characteristic functions,

$$\frac{1}{n} \sum_{j=1}^k X_j \Rightarrow \sum_{j=1}^k U_j.$$

Consequently, $\lim_{n \rightarrow \infty} P\{T_n > k\} = P\{U_1 + \cdots + U_k < 1\} = 1/k!$, after symmetry considerations. Since $P\{T_n = k\} = P\{T_n > k-1\} - P\{T_n > k\}$ the result follows.