6.4. The first part follows directly from the Radon–Nikodým theorem, and there is nothing to prove. For the second part note that whenever $B \in \mathscr{B}(\mathbf{R})$ is Lebesgue-zero, then so is $\mathbf{R} \times B \in \mathscr{B}(\mathbf{R}^2)$. Therefore, X and Y have also absolutely continuous distributions, whence follows the second part. For the third part note that X and Y are independent iff for all bounded, continuous functions ϕ_1 and ϕ_2 on $\mathbf{R}$,

$$\iint \phi_1(x)\phi_2(y) f(x,y)\, dx\, dy = \int \phi_1(x) f_X(x)\, dx \cdot \int \phi_2(y) f_Y(y)\, dy.$$

If $f(x,y) = f_X(x) f_Y(y)$ for almost all $(x,y) \in \mathbf{R}^2$, then the preceding equality holds, and so X and Y are independent. By the proof of Problem **6.1**, for all bounded, measurable $\phi : \mathbf{R}^2 \to \mathbf{R}^2$,

$$\iint \phi(x,y) f(x,y)\, dx\, dy = \iint \phi(x,y) f_X(x) f_Y(y)\, dx\, dy.$$

Apply this with $\phi(x,y)$ denoting the indicator that $|f(x,y) - f_X(x) f_Y(y)| > \varepsilon$ and then let $\varepsilon \downarrow 0$ to find that $f(x,y) = f_X(x) f_Y(y)$ for almost all $(x,y) \in \mathbf{R}^2$.