

*Solution 2.* Here is an elegant solution, due to Z. Horváth. It relies on the following real-variable lemma.

**Lemma:** *If  $a_n \rightarrow \mu$  and  $b_n \geq 0$  satisfy  $\sum_{i=1}^n b_i \rightarrow \infty$ , then  $\sum_{i=1}^n a_i b_i \sim \mu \sum_{i=1}^n b_i$ .*

**Proof** Fix  $\varepsilon > 0$ , and find  $n_0$  so large that  $|a_i - \mu| \leq \varepsilon$  for all  $i \geq n_0$ . Then,

$$\sum_{i=1}^n a_i b_i \sim \sum_{i=n_0}^n a_i b_i = (\mu \pm \varepsilon) \sum_{i=n_0}^n b_i \sim (\mu \pm \varepsilon) \sum_{i=1}^n b_i,$$

notation being clear. □

Now let  $S_0 = 0$ , and  $S_n = \sum_{j=1}^n S_j$  ( $n \geq 1$ ), so that

$$\sum_{i=1}^n \frac{X_i}{i} = \sum_{i=1}^n (S_i - S_{i-1}) \frac{1}{i} = \sum_{i=1}^n S_i \frac{1}{i} - \sum_{i=1}^n S_{i-1} \frac{1}{i} = \sum_{i=1}^n S_i \frac{1}{i} - \sum_{i=2}^{n-1} S_i \frac{1}{i+1} = S_1 + \sum_{i=2}^{n-1} S_i \left( \frac{1}{i} - \frac{1}{i+1} \right) - \frac{S_n}{n+1}.$$

By the strong law,  $S_n/(n+1) \rightarrow \mu$  a.s. Therefore,

$$\frac{1}{\ln n} \sum_{i=1}^n \frac{X_i}{i} \sim \frac{1}{\ln n} \sum_{i=2}^{n-1} \frac{S_i}{i(i+1)} \sim \frac{1}{\ln n} \sum_{i=2}^{n-1} \frac{\mu}{i+1} \rightarrow \mu \quad \text{a.s.}$$