6.29. *Solution 1.* We will need the following **Lemma:** *As* $n \to \infty$, $\sum_{i=1}^n (1/i) \sim \ln(n)$.

Proof This follows from the integral test of calculus, because $\int_1^n (dx/x) \le \sum_{i=1}^n (1/i) \le 1 + \int_1^n (dx/x)$. □

Now let us first assume that $\mathrm{E}[X_1] = 0$. Define $X_i' = X_i \mathbf{1}_{\{|X_i| \le i\}}$. Also define $S_n = \sum_{i=1}^n (X_i/i)$ and $S_n' = \sum_{i=1}^n (X_i'/i)$. By the Kolmogorov maximal inequality,

$$\mathrm{P}\left\{\max_{1\le k\le n} \left|S_k' - \mathrm{E}[S_k']\right| \ge \varepsilon \ln n\right\} \le \frac{\mathrm{Var} S_n'}{\varepsilon^2 (\ln n)^2} \le \frac{\|S_n'\|_2^2}{\varepsilon^2 \ln^2(n)} = \frac{1}{\varepsilon^2 \ln^2(n)} \sum_{i=1}^n \frac{\mathrm{E}\left[X_1^2; |X_1| \le i\right]}{i^2}.$$

Therefore,

$$\sum_{n=1}^\infty \mathrm{P}\left\{\max_{1\le k\le 2^{2^n}} \left|S_k' - \mathrm{E}[S_k']\right| \ge \varepsilon \ln\left(2^{2^n}\right)\right\} \le A \sum_{n=1}^\infty \frac{1}{4^n} \sum_{i=1}^{2^n} \frac{\mathrm{E}\left[X_1^2; |X_1| \le i\right]}{i^2} \le A \sum_{i=1}^\infty \frac{\mathrm{E}\left[X_1^2; |X_1| \le i\right]}{i^2} \sum_{n \ge \log_2(i)} \frac{1}{4^n}$$
$$\le B \sum_{i=1}^\infty \frac{\mathrm{E}\left[X_1^2; |X_1| \le i\right]}{i^2} = B\mathrm{E}\left[X_1^2 \sum_{i \ge |X_1|} \frac{1}{i^2}\right] \le C\mathrm{E}X_1.$$

By the Borel–Cantelli lemma, a.s., for all n large,

$$\max_{1\le k\le 2^{2^n}} \left|S_k' - \mathrm{E}[S_k']\right| \le \varepsilon \ln\left(2^{2^n}\right).$$

Any m is between some 2^{2^n} and $2^{2^{n+1}}$. Therefore,

$$\left|S_m' - \mathrm{E}[S_m']\right| \le \max_{1\le k\le 2^{2^{n+1}}} \left|S_k' - \mathrm{E}[S_k']\right| \le \varepsilon \ln\left(2^{2^{n+1}}\right) = 2\varepsilon \ln(2) \cdot 2^n \le 2\varepsilon \ln(2) \ln m.$$

This proves that a.s., $|S_m' - \mathrm{E}S_m'|/\ln(m) \to 0$. But $\sum_k \mathrm{P}\{S_k \ne S_k'\} = \sum_k \mathrm{P}\{|X_1| > k\} < \infty$ because $\|X_1\|_1 < \infty$. So almost surely, $S_k = S_k'$ for all k large. It suffices to prove that $|\mathrm{E}S_m'|/\ln m \to 0$. But

$$\left|\mathrm{E}S_m'\right| = \left|\sum_{i=1}^m \frac{\mathrm{E}[X_1; |X_1| \le i]}{i}\right| = \left|\sum_{i=1}^m \frac{\mathrm{E}[X_1; |X_1| > i]}{i}\right|,$$

since $\mathrm{E}X_1 = 0$. Thus,

$$\left|\mathrm{E}S_m'\right| \le \sum_{i=1}^m \frac{\mathrm{E}[|X_1|; |X_1| > i]}{i}.$$

For all $\eta > 0$, there exists i_0 such that for al $i \ge i_0$, $\mathrm{E}\{|X_1|; |X_1| > i\} \le \eta$. Therefore, $\sum_{i=i_0}^m (\cdots) \le \eta \sum_{i=1}^m (1/i) \sim \eta \ln m$. But $\sum_{i=1}^{i_0} (\cdots) \le i_0 \|X_1\|_1$. Therefore, $\limsup_m |\mathrm{E}S_m'|/\ln m \le \eta$ for all η, whence the result.

If $\mu = \mathrm{E}X_1 \ne 0$, then consider instead $X_i^\circ := X_i - \mu$. The preceding proves that $\ln^{-1}(n) \sum_{i=1}^n (X_i^\circ/i) \to 0$ a.s. Equivalently, $\ln^{-1}(n)\{\sum_{i=1}^n (X_i/i) - \mu \sum_{i=1}^n (1/i)\} \to 0$. Because $\sum_{i=1}^n (1/i) \sim \ln(n)$, we obtain the result in general.