6.17. Because X is a.s. integer-valued, we can write

$$\sum_{i=1}^{\infty} \mathbf{1}_{\{X \geq i\}} = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \mathbf{1}_{\{X=j\}} = \sum_{j=0}^{\infty} j \mathbf{1}_{\{X=j\}} = X \qquad \text{a.s.}$$

Take expectations to finish the derivation of the first claim. A useful, but equivalent, formulation is that when X is $\mathbf{Z}_+$-valued then $\mathrm{E}X = \sum_{n=0}^{\infty} \mathrm{P}\{X > n\}$. This is the variant we use below.

1. Note that for all $i \in \{2, \ldots, k\}$,

$$\mathrm{P}\{X(k) > i\} = 1 \cdot \left(1 - \frac{1}{2k-1}\right) \cdot \left(1 - \frac{1}{2k-2}\right) \cdots \left(1 - \frac{1}{2k-i+1}\right) = \frac{2k-2}{2k-1} \cdot \frac{2k-3}{2k-2} \cdots \frac{2k-i}{2k-i+1} = \frac{2k-i}{2k-1}.$$

If $i > k$, then $\mathrm{P}\{X(k) > i\} = 0$, and yet $\mathrm{P}\{X(k) > 0\} = \mathrm{P}\{X(k) > 1\} = 1$. By these remarks and the first part,

$$\mathrm{E}[X(k)] = \sum_{i=0}^{\infty} \mathrm{P}\{X(k) > i\} = 2 + \sum_{i=2}^{k} \frac{2k-i}{2k-1} = 2 + \sum_{i=2}^{k} \left(1 - \frac{i-1}{2k-1}\right) = k - \sum_{j=1}^{k-1} \frac{j}{2k-1} = \frac{3k^2-k}{4k-2}.$$

An amusing aside: $\mathrm{E}[X(k)] \sim 3k/4$ as $k \to \infty$.

2. Let N denote the first time monotonicity fails. If $n = 0, 1$ then $\mathrm{P}\{N > n\} = 1$. If $n \geq 2$ then $\mathrm{P}\{N > n\} = \mathrm{P}\{X_1 < \cdots < X_n\} + \mathrm{P}\{X_1 > \cdots + X_n\} = 2\mathrm{P}\{X_1 < \cdots < X_n\}$, because the X_i's cannot have ties with positive probability. Since $(X_1, \ldots, X_n)$ has the same distribution as $(X_{\pi(1)}, \ldots, X_{\pi(n)})$ for all permutations π of $\{1, \ldots, n\}$, and since there must exist a permutation π such that $X_{\pi(1)} < \cdots < X_{\pi(n)}$, it follows that $\mathrm{P}\{N > n\} = 2/n!$. Hence, $\mathrm{E}N = 1 + 2\sum_{n=2}^{\infty}(1/n!) = 2e - 3$.