6.10. *(i)⇒(ii)* If $X_1 \in L^p(\mathrm{P})$ then $\sum_{n=1}^{\infty} \mathrm{P}\{|X_n| > \varepsilon n^{1/p}\} = \sum_{n=1}^{\infty} \mathrm{P}\{|X_1|^p > \varepsilon^p n\} \leq \mathrm{E}\{|X_1|^p\}/\varepsilon^p$. So by the Borel–Cantelli lemma, with probability one, $|X_n| \leq \varepsilon n^{1/p}$ for all n large. This proves that $|X_n|/n^{1/p} \to 0$ a.s.

(ii)⇒ (iii) This follows from the following real-variable **Fact.** *If $a_n/n^\rho \to 0$ for some $\rho > 0$, then* $\max_{1\leq j\leq n}(a_j/n^\rho) \to 0$.

Proof: If not, then $\max_{j\leq n} a_j > \varepsilon n^\rho$ infinitely often. But there exists n_0 such that for all $n \geq n_0$ $|a_n| \leq (\varepsilon/2)n^\rho$. So $\max_{j\leq n_0} a_j > \varepsilon n^\rho$ for infinitely-many n's, which is patently nonsense. □

(iii)⇒(i) Because (iii) implies (ii), it suffices to prove that (ii) implies (i). But this too is Borel–Canelli (as in i⇒ii).