6.1. First consider the case that $f' \geq 0$. For all $\omega \in \Omega$,

$$f(X(\omega)) = \int_0^{X(\omega)} f'(x)\,dx = \int_{-\infty}^{\infty} \mathbf{1}_{\{0 \leq x \leq X\}}(\omega) f'(x)\,dx.$$

Assuming that we can handle product-measurability issues, it follows from Fubini–Tonelli that

$$\mathrm{E}[f(X)] = \mathrm{E}\left[\int_{-\infty}^{\infty} \mathbf{1}_{\{0 \leq x \leq X\}}(\omega) f'(x)\,dx\right] = \int_0^{\infty} \mathrm{P}\{X \geq x\} f'(x)\,dx.$$

This is the desired result when $f' \geq 0$. In general, we write $f' = f'_+ - f'_-$ and define $f_{(\pm)}(x) = \int_0^x f'_\pm(z)\,dz$. The preceding development yields,

$$\mathrm{E}\left[f_{(\pm)}(X)\right] = \int_0^{\infty} f'_\pm(x) \mathrm{P}\{X \geq x\}\,dx.$$

But $f_+ - f_- = f$ (why?), whence the Problem. It suffices to prove the asserted product measurability.

Evidently, f' is a product-measurable function of (x, ω); so is $x \mapsto \mathbf{1}_{[0,\infty)}(x)$. So it suffices to prove that $(x, \omega) \mapsto \mathbf{1}_{\{X \geq x\}}(\omega)$ is product-measurable.

Define $I_n(x, \omega) = \sum_{1 \leq i < j} \mathbf{1}_{[i/n,(i+1)/n)}(x) \mathbf{1}_{[j/n,(j+1)/n)}(X(\omega))$. Each I_n is manifestly product-measurable. Therefore, so is $\mathbf{1}_{\{X > x\}}(\omega) = \lim_{n \to \infty} I_n(x, \omega)$. Therefore, $\mathbf{1}_{\{y < X \leq x\}} = \mathbf{1}_{\{y < X\}} - \mathbf{1}_{\{x < X\}}$ is measurable for all $x > y$; hence, so is $\mathbf{1}_{\{X = x\}} = \lim_{y \uparrow x} \mathbf{1}_{\{y < X \leq x\}}$. Finally, we see that $\mathbf{1}_{\{X \geq x\}}(\omega) = \mathbf{1}_{\{X > x\}}(\omega) + \mathbf{1}_{\{X = x\}}(\omega)$ is measurable.