**9.25.** If $s > 0$ and $s > w$ then we follow the hint and reflect at the first hitting time of $s$ to find that

$$\mathrm{P}\left\{S(t) > s\,,\ W(t) < w\right\} = \mathrm{P}\left\{W(t) > 2s - w\right\} = \mathrm{P}\left\{N(0\,,1) > \frac{2s-w}{\sqrt{t}}\right\}.$$

Denote the latter by $F(s\,,w)$ and let $\phi$ denote the $N(0\,,1)$ density. Then, the joint density $f_{(S(t),W(t))}$ of $(S(t)\,,W(t))$ is

$$f_{(S(t),W(t))}(s\,,w) = -\frac{\partial^2 F}{\partial s \partial w}(s\,,w) = -\frac{2}{t}\phi'\left(\frac{2s-w}{\sqrt{t}}\right) \qquad s \geq \max(w\,,0).$$

Consequently, for all $a > 0$,

$$\mathrm{P}\left\{S(t) - W(t) \leq a\right\} = -\frac{2}{t} \iint_{\substack{s>\max(w,0)\\ s-w\leq a}} \phi'\left(\frac{2s-w}{\sqrt{t}}\right) ds\,dw = -2 \iint_{\substack{s>\max(w,0)\\ s-w\leq a\sqrt{t}}} \phi'\left(2s-w\right) ds\,dw.$$

Write the integral as $\int_{-a\sqrt{t}}^{0}\int_{0}^{w+a\sqrt{t}} + \int_0^\infty \int_w^{w+a\sqrt{t}}$ and compute directly to find that

$$\mathrm{P}\left\{S(t) - W(t) \leq a\right\} = \int_{-a\sqrt{t}}^{a\sqrt{t}} \phi(x)\,dx.$$

Consequently, $S(t) - W(t) = |N(0,t)|$; i.e., $S(t) - W(t)$ has the same distribution as $|W(t)|$. In fact, Paul Lévy has proved that the process $\{S(t) - W(t)\}_{t\geq 0}$ has the same "distribution" as $\{|W(t)|\}_{t\geq 0}$ in the sense that

$$\mathrm{P}\left(\bigcap_{i=1}^{m}\left\{S(t_i) - W(t_i) \in A_i\right\}\right) = \mathrm{P}\left\{\bigcap_{i=1}^{m}\left\{W(t_i) \in A_i\right\}\right),$$

for all times $t_1, \ldots, t_m \geq 0$ and all $A_1, \ldots, A_m \in \mathcal{B}(\mathbf{R})$.

**9.26.**