9.15. Choose and fix $x \geq 0$. Then, by the Markov property,

$$\mathrm{P}\left\{\sup_{[a,b]} W = x\right\} = \int_{-\infty}^{\infty} \mathrm{P}\left\{\sup_{t\in[a,b]} (W(t) - W(a)) = x - y\right\} \mathrm{P}\{W(a) \in dy\}$$

$$= \frac{1}{\sqrt{2\pi a}} \int_0^x \mathrm{P}\left\{\sup_{[0,b-a]} W = x - y\right\} e^{-y^2/(2a)}\, dy.$$

By the reflection principle,

$$H(z) := \mathrm{P}\left\{\sup_{[0,b-a]} W \leq z\right\} = \sqrt{\frac{2}{\pi(b-a)}} \int_0^z \exp\left(-\frac{u^2}{2(b-a)}\right) du.$$

Thus, $H(z) - H(z-) = \mathrm{P}\{\sup_{[0,b-a]} W = z\} = 0$ for all z. This proves (1). To prove (2) we note that

$$\mathrm{P}\left\{\sup_{[0,a]} = \sup_{[a,b]} W\right\} = \mathrm{P}\left\{\sup_{t\in[0,a]} (W(t) - W(a)) = \sup_{t\in[a,b]} (W(t) - W(a))\right\}.$$

If $H(z) := \mathrm{P}\{\sup_{t\in[a,b]}(W(t) - W(a)) = z\}$ then, according to the Markov property, the preceding is $\mathrm{E}H(\sup_{t\in[a,b]} (W(t) - W(a))) = 0$ by (1), whence follows (2). Now we combine (1) and (2) to deduce that

$$\mathrm{P}\left\{\sup_{[0,a]} W = \sup_{[a,b]} W \text{ for some rationals } 0 \leq a < b \leq 1\right\} = 0.$$

Since W is continuous, it follows that the maximum on $[0\,,1]$ is a.s. achieved at a unique point ρ.