9.13. If $a > 0$, then $T_a > t$ if and only if $\sup_{s \le t} W(s) < a$. Therefore,

$$\mathrm{P}\left\{T_a \le t\right\} = \mathrm{P}\left\{\sup_{s \in [0,t]} W(s) > a\right\} = 1 - \mathrm{P}\left\{-a \le W(t) \le a\right\};$$

cf. the reflection principle. But $W(t) = t^{-1/2}N(0,1)$. Therefore,

$$\mathrm{P}\left\{T_a \le t\right\} = 1 - \mathrm{P}\left\{-\frac{a}{\sqrt{t}} \le N(0,1) \le \frac{a}{\sqrt{t}}\right\} = 2\mathrm{P}\left\{N(0,1) > \frac{a}{\sqrt{t}}\right\},$$

by symmetry. Therefore, the density function f_{T_a} of T_a is

$$f_{T_a}(t) = \frac{\partial}{\partial t}\mathrm{P}\{T_a \le t\} = at^{-3/2} f_{N(0,1)}\left(\frac{a}{\sqrt{t}}\right),$$

using obvious notation. In the case that $a > 0$, the form of the density of T_a follows immediately from $f_{N(0,1)}(x) = (2\pi)^{-1/2}\exp(-x^2/2)$. When $a < 0$, the density is manifestly zero.

To compute $\mathrm{E}e^{i\xi T_a}$ we first choose and fix some $\lambda > 0$ and define

$$M(t) := \exp\left(\lambda W(t) - \frac{\lambda^2}{2}t\right).$$

Then, $\{M(t \wedge T_a)\}_{t \ge 0}$ is a non-negative mean-one martingale that is a.s. bounded above by $\exp(\lambda a)$. Thanks to optional stopping,

$$\mathrm{E}\exp\left(\lambda a - \frac{\lambda^2 T_a}{2}\right) = 1.$$

That is,

$$\mathrm{E}e^{-sT_a} = \exp\left(-a\sqrt{2s}\right) \qquad {}^{\forall}s \ge 0. \tag{9.2}$$

Naiively put $s := -i\xi$ to "find" that

$$\mathrm{E}e^{i\xi T_a} = \exp\left(a\sqrt{2\xi}\right) \qquad {}^{\forall}\xi \in \mathbf{R}.$$

This is actually the correct answer. Here is a way to prove this: The left-hand side of (9.2) is analytic in $s \in \mathbf{C}$, and the right-hand side is analytic on $\{z \in \mathbf{C} : \mathrm{Re}z \ge 0\}$. Analytic continuation does the rest. In order to finish, we need to verify that $\{T_a\}_{a \ge 0}$ has i.i.d. increments. Let $a, b \ge 0$ be fixed, and note that $T_{a+b} - T_a$ is the first time the process $t \mapsto W(T_a + t) - W(T_a)$ hits b. The strong Markov property of W proves that $T_{a+b} - T_a$ is independent of $\mathcal{F}_{T_a}$ [and thence T_a], and has the same distribution as T_b.