Math 5010–001: (Summer 2010)
Solutions to Sample Midterm Three

Last update: July 13, 2010

Please read the rest only after you have seriously attempted all of the problems.

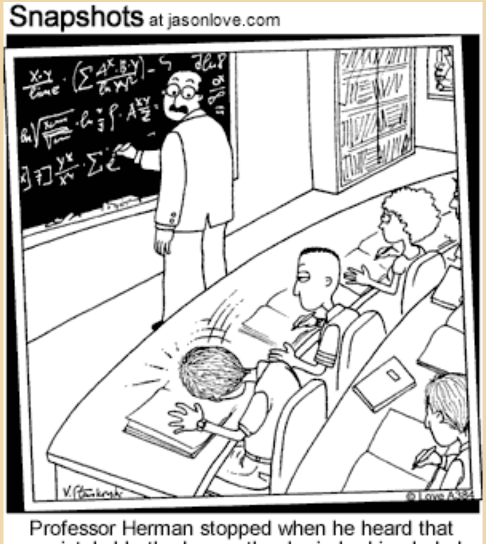
1. Suppose that, in a particular application requiring a single battery, the mean lifetime of the battery is 4 weeks with a standard deviation of 1 week. The battery is replaced by a new one when it dies, and so on. Assume lifetimes of batteries are independent. What approximately is the probability that more than 26 replacements will have to be made in a two-year period starting at the time of installation of a new battery, and not counting that new battery as a replacement? [Hint: Use a normal approximation.]

   Solution: I will assume that one *year* = 48 weeks, whence 2 *years*= 96 weeks. Let X_j denote the lifetime of replacement battery number j in weeks. We know that the X_j's are independent and identically distributed, each having [common] mean $\mu = 4$ weeks and SD $\sigma = 1$ week. We are asked to find

$$P\{X_1 + \cdots + X_{26} \le 96\} = P\left\{\frac{X_1 + \cdots + X_{26} - 26\mu}{\sqrt{26}\sigma} \le \frac{96 - 26\mu}{\sqrt{26}\sigma}\right\}.$$

   Assuming that $n = 26$ is sufficiently large, we appeal to normal approximation [the central limit theorem; see p. 196 of your text] to find that the answer is approximately

$$\Phi\left(\frac{96 - 26\mu}{\sqrt{26}\sigma}\right) \approx \Phi(-1.57) = 1 - \Phi(1.57) = 1 - 0.9418 = 0.0582.$$



Professor Herman stopped when he heard that unmistakable thud -- another brain had imploded.

2. A fair die is rolled repeatedly. Let X denote the number of rolls until the first 6 is rolled. Let Y denote the number of rolls until the first time an even number of dots are rolled.

   (a) Are X and Y independent? Justify your answer.

   Solution: No. Perhaps the easiest way to see this is to observe that $X \geq Y$. Thus, for example, $P\{X \geq 2\} > 0$, whereas $P\{X \geq 2 \mid Y = 1\} = 0$.

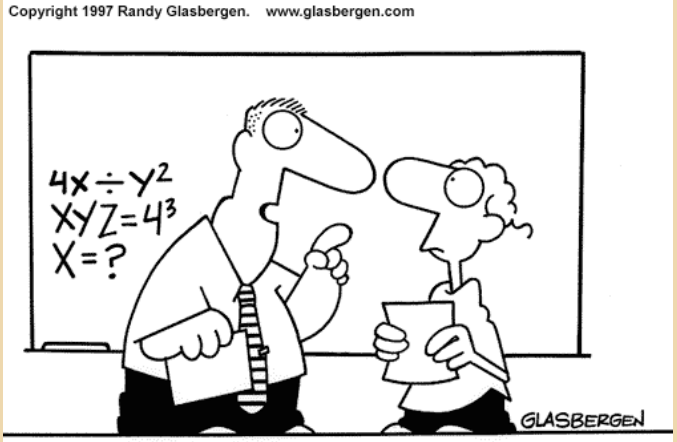
   (b) Find $P\{X = Y\}$. This is the probability that we roll 6 dots at the first instance of rolling an even number of dots.

   Solution: First, we write $P\{X = Y\} = \sum_{n=1}^{\infty} P\{X = Y = n\}$. Next we may note that the event that $X = Y = n$ is the event that: (a) The first $n-1$ tosses all result in an odd number of dots; and (b) the nth toss yields a six. Because the chances are one-half that a given toss yields an odd number of dots, $P\{X = Y = n\} = (1/2)^{n-1} \times \frac{1}{6}$. Therefore,

$$P\{X = Y\} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \frac{1}{6} = \frac{1}{6} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} = \frac{1}{3}.$$

"I think that's right, but let me check."

3. A certain population is comprised of half men and half women. Let X denote the total number of men in a random independent [i.e., with replacement] sample of 10 people from this population. Compute $E[e^{\lambda X}]$ for every real number λ.



"Algebra class will be important to you later in life because there's going to be a test six weeks from now."

Solution: We know that X has a binomial distribution, with parameter $n = 10$ and $p = \frac{1}{2}$. Therefore,

$$E\left[e^{\lambda X}\right] = \sum_{k=0}^{10} e^{\lambda k} \binom{10}{k} \left(\frac{1}{2}\right)^{10} = 2^{-10} \sum_{k=0}^{10} \binom{10}{k} e^{\lambda k}.$$

By the binomial theorem,

$$\sum_{k=0}^{n} \binom{10}{k} e^{\lambda k} = \left(1 + e^{\lambda}\right)^{10}.$$

Therefore,

$$E\left[e^{\lambda X}\right] = \left(\frac{1 + e^{\lambda}}{2}\right)^{10}.$$

4. Suppose X has a continuous distribution with density $f(x) = c/x^3$ when $x > 1$ and $f(x) = 0$ otherwise.

(a) Compute c. **Solution:** We need $\int f = 1$; therefore,

$$1 = c\int_1^\infty \frac{1}{x^3}\,dx = \frac{c}{2} \quad \Rightarrow \quad c = 2.$$

(b) Compute $P\{X \le 2\}$. **Solution:** We integrate

$$P\{X \le 2\} = \int_1^2 \frac{2}{x^3}\,dx = -x^{-2}\Big|_1^2 = \frac{3}{4}.$$

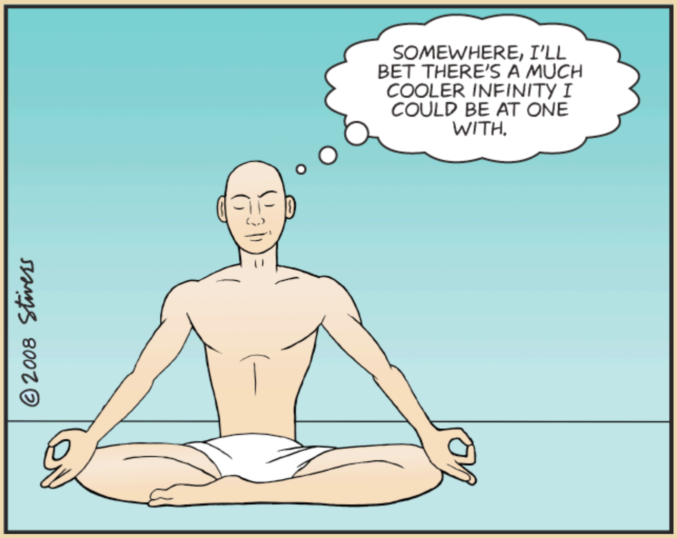
(c) Compute $E(X)$ and $\mathrm{Var}(X)$. **Solution:**

$$EX = \int_1^\infty x\frac{2}{x^3}\,dx = 2\int_1^\infty \frac{1}{x^2}\,dx = 2,$$

and

$$E(X^2) = \int_1^\infty x^2\frac{2}{x^3}\,dx = 2\int_1^\infty \frac{1}{x}\,dx = \infty.$$

Therefore, $\mathrm{Var}(X)$ is well defined, but infinite.



Good luck on Monday.