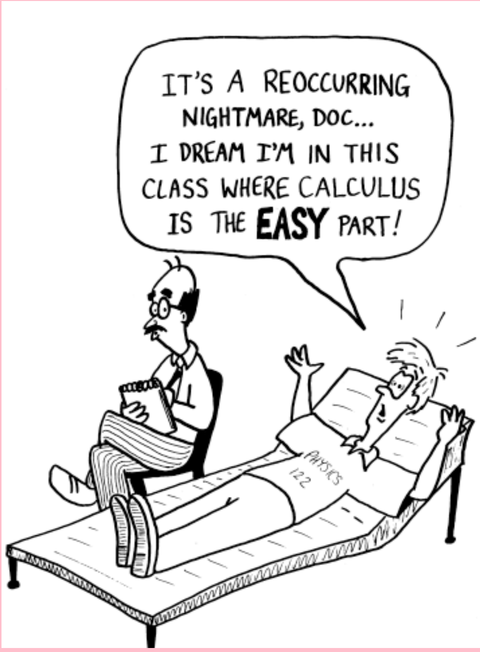
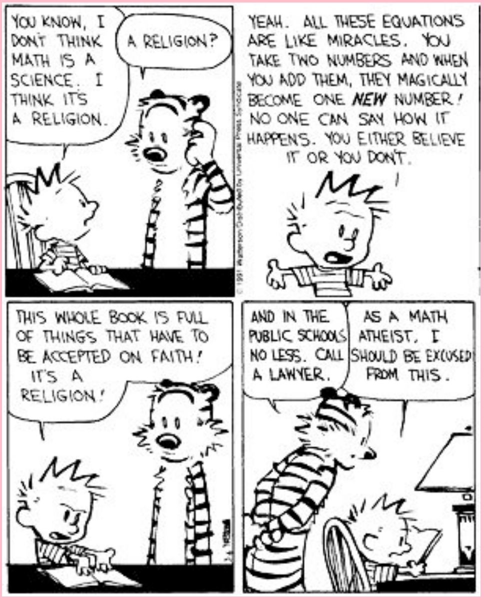
Math 5010–001: (Summer 2010)
Solutions to Sample Midterm Two

Last update: June 24, 2010



Please read the rest only after you have seriously attempted all of the problems.

1. Two fair dice are rolled at random and independently from one another. Let X_j denote the number of dots rolled by die number j and $X = \max(X_1, X_2)$. Find $P\{X = k\}$ for all k.



Solution: The possible values of X are $1, \ldots, 6$. This leads us to the following table of probabilities:

Possible value	Probability
1	1/36
2	3/36
3	5/36
4	7/36
5	9/36
6	11/36

2. The probability of being dealt a full house in a hand of poker is approximately 0.0014.

   (a) What is the probability that, in 1,000 hands of poker, you will be dealt at least 2 full houses? You may assume that the hands were dealt independently from one another.

   (b) How many hands of poker should you play [independently] so that with approximate probability 0.5 you are dealt at least 50 full houses?

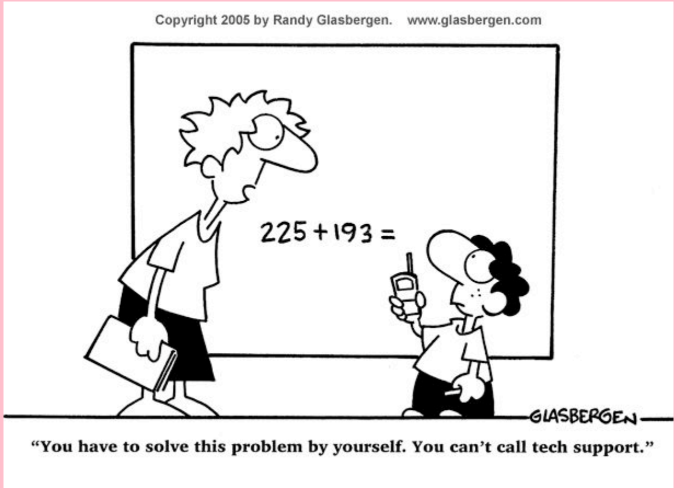
Solution to (a):

$$1 - \left[\binom{1000}{0} 0.9986^{1000} + \binom{1000}{1} 0.0014 \times 0.9986^{999}\right] \approx 0.4087475.$$

Solution to (b): Let n denote the number of hands needed. Let X be a Binomial$(n\,, 0.0014)$. Then we want n so that $P\{X \geq 50\} \approx 0.5$. Normal approximation [$\mu = 0.0014n$ and $\sigma = \sqrt{0.0014 \times 0.9986 \times n}$] tells us that

$$0.5 \approx P\{X \geq 50\} \approx 1 - \Phi\left(\frac{50-\mu}{\sigma}\right).$$

Therefore, we want $\mu = 50$, whence $n = 50/0.0014 \approx 35714.286$. That is, we need $n \geq 35715$.



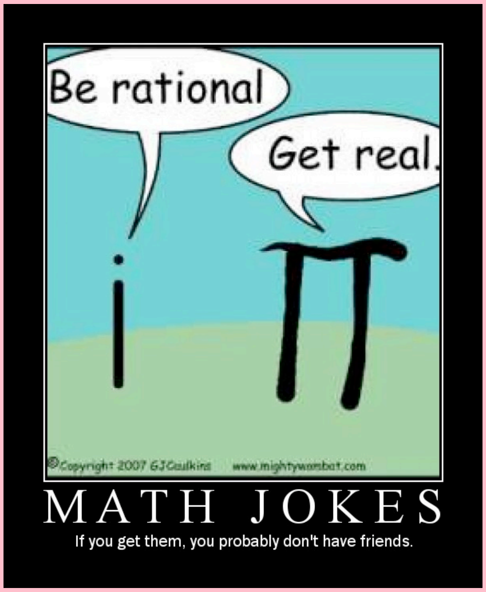
3. Let N be a fixed positive integer. You select a subset of $\{1, \ldots, N\}$, all possible subsets are equally likely.

   (a) What is the probability that the number i is in the randomly-selected subset? Answer this question for every $i = 1, \ldots, N$.

   (b) Use your answer to the preceding to find $E(X)$, where X denotes the number of elements of that randomly-selected subset.

   Solution to (a): There are 2^N subsets; 2^{N-1} of them do not have the number i in them; therefore $2^N - 2^{N-1} = 2^{N-1}$ of them have the number i in them. Consequently,

$$P\{\text{the number } i \text{ is in the random subset}\} = \frac{2^{N-1}}{2^N} = \frac{1}{2}.$$



   Solution to (b): Let A_i denote the event that i is in the subset. Then $I_{A_1} + \cdots + I_{A_n}$ is the number of elements in the random subset. I.e., X. It follows that

$$E(X) = P(A_1) + \cdots + P(A_N) = \frac{N}{2}.$$

4. I choose a number at random from 1 to N, where N is a fixed nonrandom positive integer. Your task is to guess my choice. You proceed by asking: "Is it 1? Is it 2? ..." until you find the correct number. Let X denote the number of questions you have to ask in order to find the randomly selected number.

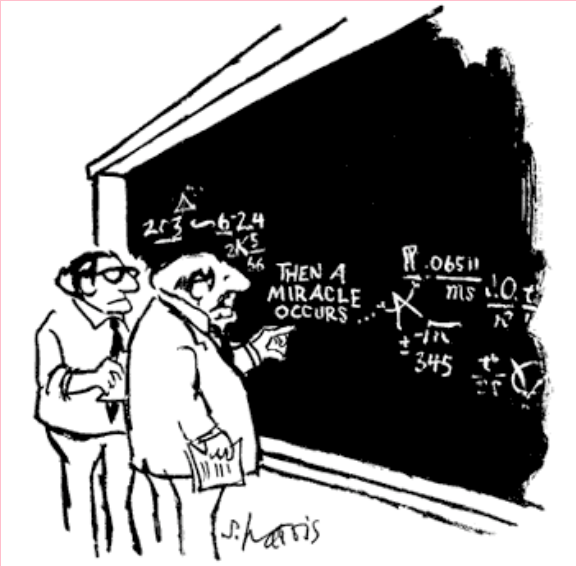
   (a) Compute $P\{X = k\}$ for every k.

   (b) Compute $E(X)$.

   Solution to (a): Some thought shows that X *is* the number that I chose. Therefore, $P\{X = k\} = 1/N$ if $k = 1, \ldots, N$.

   Solution to (b): From part (a) and the examples worked out during lecture,

$$E(X) = \frac{1 + \cdots + N}{N} = \frac{N+1}{2}.$$



"I think you should be more explicit here in step two."

Good luck on Monday. And see you in a week.